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Reinforced Concrete Arch of the Melan Type in Branch Brook Park, Newark, N. J.

PLAIN AND REINFORCED

CONCRETE ARCHES

BY

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AUTHORIZED TRANSLATION

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PREFACE

THIS book is a translation of Professor J. Melan's article on arches (*Theorie des Gewölbes und des Eisenbetongewölbes im besonderen*), published in F. von Emperger's *Handbuch für Eisenbetonbau* (Berlin, 1908). A second, revised edition of this work appeared in 1912, with much valuable material added by Melan; this latest edition is represented in this translation.

The recognized leading position of the author in this field of structural design and the extensive use of his system of arch construction in all parts of the world should be sufficient justification for presenting this book to the engineers of this country.

The present work constitutes one of the most thorough treatments of reinforced concrete arches in any language. After a brief discussion of the fundamental principles of arches and a simple though comprehensive treatment of the stresses in reinforced concrete sections, there are presented analytic and graphic methods for the complete design of all types of concrete arches occurring in practice. The graphic methods which are given, permitting the use of influence lines, will be found very practical, although new to American designers. The effects of temperature, of yielding abutments and of non-vertical loads are separately considered.

The following are some special features which should commend this book to American readers. In addition to the exact treatments, simple approximations and short cuts are introduced which will prove highly useful for preliminary and less exacting designs. Easily-applied formulæ are developed for determining in advance the best curve for an arch and

the required dimensions and reinforcement. These various methods are illustrated with numerical examples borrowed from actual practice. The principle of the Melan system of arch construction is fully explained and its inherent economy concretely demonstrated. Complete computations of stresses, analytic and graphic respectively, of two notable examples of reinforced concrete arches are given in the appendix.

In the closing chapter there is introduced an expeditious method for proportioning or investigating concrete sections with single or double reinforcement by means of a chart. This diagram has been taken from Prof. Melan's *Hilfstafel zur Berechnung doppelt-armierter Balken, Platten und Gewölbe* (Prague, 1907). An additional set of curves (in red) has been added by the translator and a key with numerical examples has been appended in order to enhance the usefulness of this chart.

In making this translation, the notation has been modified and an index of the symbols inserted for greater clearness. All of the numerical examples and data have been converted from metric to English units. With the exception of a few minor annotations, the original text (second, revised edition) has been faithfully followed throughout.

The translator desires to express his indebtedness to Professor Melan for his generous consent to the publication of this translation and for helpful courtesies extended during its preparation.

D. B. S.

NEW YORK, January, 1915.

CONTENTS

EXAMPLES

INDEX TO NOTATION

A = area of cross-section.
A_0 = mean area of cross-section. (Page 42.)
A_s = area of steel reinforcement per width b.
A, B = vertical components of end-reactions. (Fig. 6.)
A_1, A_2, A_3 = coefficients defined by Eqs. (20).
a = abscissa of any load P. (Figs. 6, 13.)
a_0 = ratio of reinforcement at the crown. (Chap. XIII.)
a_1, a_2 = distances of extreme fibers from center of gravity of section. (Fig. 2.)
a', a'' = ratios of tension and compression reinforcement. (Chap. XIV.)

b = width of section.

C, C', C'' = arch functions introduced in Eq. (68).
c = small correction term. (Page 47.)

d = radial depth of section. (Fig. 4.)
d_0 = thickness of arch at crown. (Fig. 34.)
d_1 = thickness of arch at quarter-points. (Fig. 38.)
d_0' = depth at crown reduced to equivalent in plain concrete. (Eq. (92).)
d_1' = depth at quarter-points reduced to equivalent in plain concrete. (Eq. (97).)

E = modulus of elasticity. (E_s, for steel; E_c, for concrete; E_{ct}, for concrete in tension; E_{cp}, for concrete in compression.)
e = distance between axis of steel and axis of concrete $= e_1 + e_2$. (Fig. 4.)

f = versed-sine or rise of arch-axis. (Fig. 6.)
f' = rise of line of resistance. (Eq. (46).)
f_0 = reduced arch-rise, defined by Eq. (80).
f_1, f_2 = stresses in extreme concrete fibers. (Fig. 3.)
f_1', f_2' = stresses in extreme steel fibers. (Fig. 5.)
f_u = fiber stress at unit distance from neutral axis.
f_c, f_{ct} = extreme fiber stresses in the concrete.

f_s, f_{sp}, f_{st} = extreme fiber stresses in the steel reinforcement.
f_t = fiber stress produced by temperature change.

G = a concentrated vertical load. (Fig. 17).
g = intensity of dead load per unit length.
g_0 = intensity of dead load at the crown.
g' = intensity of dead load above the crown $= \gamma_1 u_0$. (Chap. XIII.)
g_0' = dead load at crown carried by the centering. (Chap. XIII.)

H = horizontal thrust of arch = horizontal component of R. (Fig. 1.)
H_g = horizontal thrust produced by dead load g. (Fig. 24.)
H_p = horizontal thrust produced by live load p covering the whole span. (Fig. 24.)
H_1 = horizontal thrust for loading on one-half of the span.
H_t, M_t, etc. = effects of temperature. (Chap. V.)
H' = axial thrust at crown of arch. (Fig. 22.)
H_0, H_0', H_0'' = Horizontal thrusts in continuous arches on assumption of rigid supports. (Fig. 33.)
h = vertical intercept between line of resistance and arch-axis. (Fig. 15.)
h_a, h_b = heights of elastic end piers. (Fig. 26.)
h_1, h_2 = heights of piers. (Fig. 33.)
h' = difference in elevation of ends of arch. (Fig. 20.)
h = altitude of roadway parabola. (Fig. 34.) (Chap. XI only.)

I = moment of inertia of section.
I_s = moment of inertia of steel reinforcement.
I_0 = value of I at crown. (Chap. VII.)
ι = sectional constant $= I \div bd^3$. (Chap. XIV.)

j = distance of axis of the steel from the compression fibers of a section. (Fig. 5.)

k = fraction of weight of arch carried by the centering. (Chap. XIII.)
k_1, k_2 = lower and upper core-distances. (Fig. 2.)

l = span of arch-axis. (Fig. 6.)
l_1, l_2 = abscissæ of ends of arch. (Fig. 27.)

$\mathfrak{M}$ = summation of moments, introduced in Eq. (16).
M = bending moment at any section, referred to its center axis.
M_a = bending moment at any section, referred to its resultant axis.
M_0 = bending moment at the crown. (Fig. 13.)
M_b = bending moment in a simple beam.
M_c = moment of vertical loads about the crown. (Eq. (9).)
M' = live-load bending moment in loaded half of arch.
M'' = live-load bending moment in unloaded half-arch.

m = ratio of elastic moduli of concrete in tension and compression $= E_{ct} : E_{cp}$.
m, m' = coefficients of resisting moment. (Chap. XIV.)

N = the normal force acting on a section, = normal component of R. (Fig. 2.)
n = ratio of elastic moduli of steel and concrete $= E_s : E$

$\mathfrak{P}$ = summation of loads, introduced in Eq. (16).
P = a concentrated load acting on arch. (Fig. 1.)
p = intensity of live load.
p_1 = intensity of live load assumed over one-half of the span.
p_n = lever arm of N about center of gravity of section. (Fig. 5.)
p_0 = pole distance. (Fig. 25.)

q = total intensity of loading at any point.
q_0 = total intensity of loading at crown. (Eq. (70).)
q_1 = total intensity of loading at springing. (Fig. 36.)
q' = correction term introduced in Eq. (27).
q'' = abbreviation defined by Eq. (71).

R = resultant pressure at any section. (Fig. 2.)
r = radius of curvature of arch-axis at any section. (Fig. 37.)
r_0 = radius of curvature of line of resistance at the crown. (Fig. 37.)
r' = radius of curvature of the intrados at the crown. (Fig. 37.)

S = transverse shear, = radial component of R. (Fig. 2.)
S_b = transverse shear in a simple beam.
s = maximum fiber stress at a section (plain concrete).
s_0 = average intensity of stress in the concrete at the crown.
ds = element of arch-axis.

T = change of temperature.
t, t_0, t_1, t_2 = intercepts between arch-chord and X-axis. (Figs. 16, 19.)

u = depth of filling at any point. (Fig. 34.)
u_0 = depth of filling above crown. (Fig. 34.)
u, v = coordinates of center of curvature of arch axis. (Fig. 15.)

V = vertical shear at crown. (Fig. 25.)

W = concentrated load applied parallel to X-axis. (Fig. 22.) (Chap. VI.)
W = total work of elastic deformation of the arch.
W, W', W'' = elastic weights (concentrations), defined by Eqs. (36).
w, w', w'' = elastic weights (distributed), defined by Eqs. (33).
w_1 = abbreviation introduced on page 90. (Chap. XI.)
w_2 = abbreviation defined by Eq. (86).

X_1, X_2 = reaction unknowns defined by Eqs. (28).
x, y = coordinates of arch-axis. (Fig. 6.)
x' = abscissæ of arch-axis measured from right end of span. (Fig. 16.)
x = distance of neutral axis from the compression fibers of a section. (Fig. 5.) (Chap. II only.)
x = fractional distance of neutral axis from the compression fibers. (Chap. XIV only.)

y' = ordinates of arch-axis above the arch-chord. (Fig. 16.)
y_1 = ordinate at quarter-point of arch. (Fig. 36.)

Z = section modulus. (Chap. XIII.)
z_0, z_1, z_2 = ordinates of chord of line of resistance. (Fig. 15.)
z, z', z'' = ordinates of influence lines for H, X_1, X_2. (Fig. 17.)

α = inclination of arch-cord to the horizontal (Fig. 6.)
α, β, ϕ, ϵ = coefficients defined by Eqs. (19).
α_r, β_r, ϕ_r, γ_r, δ_r = coefficients defined by Eqs. (42).
β = constant of roadway parabola $=\dfrac{4h}{l^2}$. (Fig. 34.)
γ, γ_1 = densities of concrete and spandrel filling, respectively.
γ' = density of concrete multiplied by $\dfrac{k}{1+na_0}$. (Eq. (94).)
δ_1, δ_2 = horizontal displacements of abutments.
δ_A = horizontal displacement produced by $H=1$ acting on arch.
δ_P = horizontal displacement produced by $H=1$ acting on top of pier.
δ_a, δ_b = horizontal displacements of abutments produced by $H=1$.
ϕ_a, ϕ_b = rotation of end sections produced by $H=1$.
ψ_a, ψ_b = rotation of end sections produced by $M=1$.
ϕ = inclination of section to the vertical. (Fig. 6.)
ψ_A = end rotation producible by $M=1$ acting on arch.
ψ_P = rotation of top of pier producible by $M=1$.
$d\psi$ = angular rotation of any section. (Fig. 14.)
τ = inclination of axis of abscissæ. (Fig. 15.)
ω = coefficient of linear expansion.

PLAIN AND REINFORCED CONCRETE ARCHES

CHAPTER I

THE FORCES ACTING ON AN ARCH

CONSIDER a strip cut from a cylindrical arch by two vertical planes parallel to the heads of the arch, with no normal or shearing forces acting in the planes of section. Strictly, the last condition can be realized only in an arch of homogeneous material, loaded uniformly in the transverse direction; furthermore, the heads of the arch must be perpendicular to the elements of the cylindrical surface or soffit, i.e., a right arch is presupposed. In all other cases, particularly in skew-arches, the above assumption can be only approximately correct.

On the arch-strip thus abstracted, we have certain forces acting; usually these are simply forces of gravity, so that we may confine our attention to vertical loading. These loads consist of the dead weight of the arch and superstructure, together with the useful or live load. Now consider the arch-strip (Fig. 1) divided into separate segments or voussoirs by joint-planes or sections 1, 2, 3, . . . , with the end-planes A and B resting against fixed abutments. With the previous assumption of no shears in the head-planes, each voussoir, e.g., 1–2, is subjected to the action of three forces, namely, the given external load P_2 and the two resultant pressures R_1 and R_2 in the abutting joints. These forces must hold each other in equilibrium and may therefore be represented by a

force-triangle 1–2–O; in this way either joint-pressure may be determined when the other is known. Consequently all of the joint-pressures may be found as soon as any one of these forces is given in its magnitude, direction and point of application.

The forces acting at the end-planes A and B are the abutment-pressures, and their opposing forces are the abutment-reactions K_1 and K_2. The application of these reactions takes the place of the abutments, so that the arch may be con-

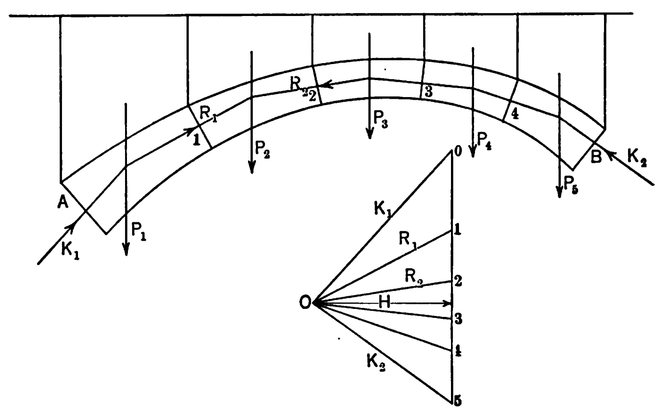

FIG. 1.—The Forces Acting on an Arch. The Line of Resistance.

sidered an independent system in equilibrium under the forces ΣP, K_1, and K_2.

The action and reaction between adjacent voussoirs occur along the lines of the forces R; therefore the polygon composed of these forces is appropriately named the *Line of Resistance.* It may be obtained as the funicular polygon of the forces P with the end-reactions K as the terminal sides.

The line of resistance changes with the number and direction of the joint-planes. The latter should be so chosen that the joint-pressures may be as nearly as possible normal to the respective joints, for then no possibility of sliding need be considered. We comply with this requirement by placing the joints normal to the axis of the arch. However, a change

in the assumed direction of the joint-planes in constructing the line of resistance will not alter the latter much; so that for a simpler, approximate construction, the joints may be assumed vertical instead of radial.

If the number of joints be increased by interpolating others, the number of sides of the resistance-polygon will also be increased; and if the joints be taken infinitely close together, the line of resistance becomes a continuous curve. It is immaterial whether the joints are real divisions, as in a voussoir arch, or only imaginary sections normal to the axis of the arch.

By the *Line of Pressure* of an arch is meant the geometric locus of the points of application of the joint-pressures, i.e., their points of intersection with the respective joints, real or assumed. Resistance-line and pressure-line are commonly identified. Practically, this is always permissible; but a mathematical coincidence of the two curves can never occur except with vertical joints.

If the loading is all vertical, the horizontal components of all the joint-pressures R are equal to each other and to the horizontal component of the end-reactions. The latter is named the *Horizontal Thrust* (H) of the arch. The line of resistance is then given as the funicular polygon or equilibrium curve of the loads P, constructed with H as the pole-distance and drawn in its proper position relative to the arch, namely so that the terminal sides coincide with the end-reactions. In order to locate the line thus specified, three conditions must in general be supplied. These may be: the magnitude, direction, and line of action of any one of the joint-pressures or end-reactions; or else three points through which the required line will have to pass as fixed by means of hinges. The three-hinged arch is therefore the only form which is statically determinate. In all other forms, an infinite number of equilibrium polygons can be drawn for any loading, only one of which can represent the true line of resistance; to determine this line, the laws of deformation of the arch must be known.

CHAPTER II

THE INTERNAL STRESSES IN AN ARCH

THE line of resistance, with the joint-pressures known, serves to determine the internal stresses in the individual joints or sections of the arch; the normal or compressive stresses are found from N, the component of the pressure normal to the joint, and the shearing stresses from S, the component parallel to the joint. The latter component is always very small if the arch-curve is properly designed to fit the loading, and may generally be disregarded in concrete arches, since the relatively high dead load prevents any large variation in the line of resistance.

The distribution of the normal stresses in any cross-section may be figured according to the common laws of elastic materials. It has been demonstrated by tests and observations on actual structures that building stones and concrete exhibit the properties of elasticity, although not so perfectly as to permit a constant ratio between stress and strain to be assumed for all values of stress. For *compression* within the limits of safe stress, however, such proportionality may be assumed without any considerable error; so that a constant coefficient of elasticity may be used in the calculations. The *theory of the elastic arch* may therefore be applied to arches of all classes of masonry, including monolithic arches of concrete, provided no tensile stresses or only very small tensile stresses are allowed to occur. The same theory also forms the basis for the design of reinforced concrete arches, which will be considered later in more detail.

Let A = the area of any section of an arch;

I = its moment of inertia about the gravity axis perpendicular to the force-plane;

a_1, a_2 = the distance of the upper or lower extreme fibers from the center of gravity, respectively;

N = the axial thrust, i.e., the force acting perpendicular to the cross-section;

M = the moment of the joint-pressure R about the gravity axis of the section.

Then, for *straight* ribs (or beams), the extreme fiber stresses are given by the familiar relation,

$$f_{1,2} = \frac{N}{A} \pm \frac{M \cdot a_{1,2}}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For *curved* ribs, with radius of curvature $= r$, the more exact formula gives

$$f_{1,2} = \frac{N}{A} + \frac{M}{A \cdot r} \pm \frac{M \cdot r \cdot a_{1,2}}{I(r \pm a_{1,2})}, \quad . \quad . \quad . \quad (1a)$$

or, with close approximation,

$$f_{1,2} = \frac{N}{A} + \frac{M}{A \cdot r} \pm \frac{M \cdot a_{1,2}}{I}. \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

Only when the radius of curvature is small in proportion to the sectional dimensions, need this formula be employed. For calculating the stresses in ordinary arches, formula (1) for straight beams is sufficiently accurate.

The moment M may be expressed (see Fig. 2) by

$$M = R \cdot z = N \cdot p_n = H \cdot h.$$

Hence, by Eq. (1)

$$f_{1,2} = \frac{N}{A}\left(1 \pm \frac{p_n \cdot A \cdot a_{1,2}}{I}\right)$$

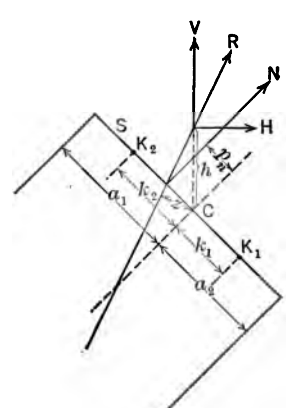

FIG. 2.—Internal Stresses.

or, introducing the core-distances

$$k_1=\frac{I}{A\cdot a_1} \quad \text{and} \quad k_2=\frac{I}{A\cdot a_2},$$

we have

$$f_1=\frac{N}{A}\left(1+\frac{p_n}{k_1}\right), \quad f_2=\frac{N}{A}\left(1-\frac{p_n}{k_2}\right).$$

Consequently, in order that f_1 and f_2 may have positive signs, representing stresses of compression, we must have $k_2>p_n>-k_1$, i.e., the point of application of the resultant joint-pressure must fall between the two core-points of the cross-section. Hence *no tensile stresses will occur anywhere in the arch if the line of resistance lies wholly within the core-area of the arch.*

If the cross-section of the strip of arch, of homogeneous material, is a rectangle of unit width and depth d, Eq. (1) gives

$$f_{1,2}=\frac{N}{d}\pm\frac{6M}{d^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The core-points fall at the one-third points of the depth of the section and, for such an arch, the line of resistance must lie within the *middle third* of the depth if no tensile stresses are to occur.

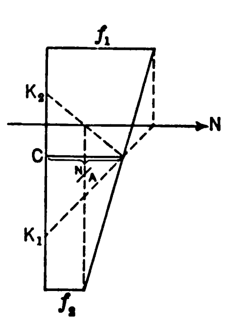

FIG. 3.—The Distribution of Normal Stresses. Graphic Method.

The distribution of normal stresses in the cross-section of an arch may thus be found by the common rules for eccentric loading in material of constant elastic modulus; and for this purpose we may employ the familiar graphic construction (Fig. 3): Erect the ordinate $\frac{N}{A}$ as the stress at the center of gravity, and draw straight lines from the core-points K_1 and K_2 through the extremity of this ordinate; these two lines will intercept on

the line of action of the normal thrust N the lengths representing to scale the extreme fiber stresses f_1 and f_2.*

If the material is not of homogeneous elasticity, retaining the assumption that the sections of the arch remain *plane* after deformation, the above constructed linear variation in the cross-section will no longer represent the normal stresses themselves, but quantities equal to $\frac{1}{E}$ times those stresses. This relation, as is well known, forms an important element in the theory of composite members of concrete and steel.

In calculating the stresses in a *reinforced concrete arch*, this theory will be applied. If only compressive stresses occur in any cross-section, we may adopt a constant coefficient of elasticity for the concrete; and for the steel we introduce the customary mean ratio $n = E_s : E_c = 15$. (In well-hardened concrete, $n = 10$.) This treatment, based upon a uniform effectiveness of the entire section of the concrete, may also be applied even if there occur slight tensile stresses for which the elastic modulus is not sensibly diminished. Let us designate this condition of the composite member as *Phase I*.

This method of design, however, will no longer give correct results when there appear in the concrete larger tensile stresses than that material can possibly withstand. For such a case the treatment which has met with general acceptance in reinforced concrete practice is based on a so-called *Phase II;* this assumes that the concrete cannot stand any tension; therefore, in the portion of the cross-section subjected to such stress, the concrete is considered removed and the steel reinforcement is made to resist the entire tension. Since, however, the concrete will take a certain amount of tension up to the point of actual failure, this method of design yields

* *Proof for the Construction in Fig. 3.* (—D. B. S.)
By similar triangles,

$$f_1 : \frac{N}{A} = (k_1 + p_n) : k_1, \quad \therefore f_1 = \frac{N}{A}\left(1 + \frac{p_n}{k_1}\right);$$

$$f_2 : \frac{N}{A} = (k_2 - p_n) : k_2, \quad \therefore f_2 = \frac{N}{A}\left(1 - \frac{p_n}{k_2}\right). \qquad \text{Q.E.D.}$$

results that are somewhat too severe, namely larger compression in the concrete and larger tension in the steel than will actually occur. Furthermore, the method affords no conclusion as to the actual tensile stresses in the concrete; the latter should be known, at least approximately, in order to render possible an estimate of the safety of the material against the formation of tensile cracks. We are thus led to a third method of design which considers the concrete as taking tension but with a reduced value of the elastic coefficient, E_{ct}. The latter, to be accurate, should be made to vary with the magnitude of the stresses; but in order to obtain simple formulæ for calculation let us assume the coefficient of elasticity for concrete in tension E_{ct} as constant over the entire tensile section and write this as m times the coefficient for compression: $E_{ct} = m \cdot E_{cp}$. For large tensile stresses, near the ultimate resistance, a proper value to use is $m = 0.3$ to 0.4. We thus obtain values for the stresses which agree very fairly with the results of experience and special tests. With $m = 0$, this method of calculation reduces to the simple form of Phase II; with $m = 1$, it passes into Phase I.

In reinforced concrete arches with correct form of curve and adequate depth of section, the predominating stresses will be those of compression, and only small tensile stresses will apear. Hence it is usually permissible and sufficient to calculate the stresses by Phase I. Nevertheless, considerable tensile stresses may occur at certain points under severe conditions of loading, particularly under the action of temperature changes. In such case, the steel reinforcement permits a smaller depth to be used than would be required in masonry or plain concrete arches, since in the latter it is imperative that all tensile stresses be avoided or at least kept within very small limits. However, even in reinforced concrete arches, tensile stresses should be permitted only up to a value at which no formation of cracks need be feared. Consequently, if the computation by Phase I yields values for the tensile stresses in the concrete at any sections exceeding the resistance of that material, such sections should be reinvestigated by the

methods of Phase II: assuming, for the sake of safety, $m=0$ for finding the maximum stresses in the steel and the maximum *compression* in the concrete, and $m=0.4$ for finding the probable *tension* in the concrete; the latter should not exceed the ultimate resistance.

The necessary formulæ for the above methods of design will now be developed.

1. Design by Phase I

We have:

Concrete cross-section, width of arch-strip $=b$,
depth or thickness of rib $=d$,
Area of steel per width b $=A_s$,
Ratio of elastic coefficients, $E_s : E_c$ $=n$,
Equivalent combined cross-section (i.e., total section reduced to the elasticity of the concrete):

$$A=b\cdot d+n\cdot A_s. \qquad (3)$$

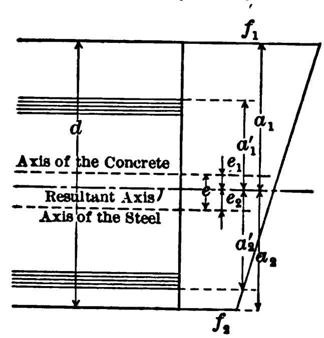

Fig. 4.—Reinforced Concrete Section. Phase I.

Distance of the center of gravity of the reinforcement from the axis of the arch $=e$,
Position of the resultant center of gravity of the combined cross-section (see Fig. 4):

$$\begin{cases} e_1=\dfrac{n\cdot A_s}{A}\cdot e. \\ \\ e_2=\dfrac{bd}{A}\cdot e. \end{cases}$$

Moment of inertia of the concrete section about its principal axis $=\frac{1}{12}bd^3$,

Moment of inertia of the steel section about its axis of gravity $=I_s$.

Equivalent moment of inertia of the combined cross-section about the resultant axis of gravity:

$$I=\tfrac{1}{12}bd^3+bde_1{}^2+n(I_s+A_se_2{}^2). \quad . \quad . \quad . \quad . \quad (4)$$

On the cross-section of breadth b, let there be acting a joint pressure whose axial component is N and whose moment about the resultant gravity axis is M_a. Then, if a_1 and a_2 are the distances of the extreme fibers of the concrete from that axis, and a'_1 and a'_2 the corresponding distances for the steel reinforcement, the extreme fiber stresses in the concrete will be

$$\left.\begin{aligned} f_1&=\frac{N}{A}+\frac{M_a\cdot a_1}{I} \\ f_2&=\frac{N}{A}-\frac{M_a\cdot a_2}{I} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and those in the steel will be

$$\left.\begin{aligned} f'_1&=n\left(\frac{N}{A}+\frac{M_a}{I}\cdot a'_1\right) \\ f'_2&=n\left(\frac{N}{A}-\frac{M_a}{I}\cdot a'_2\right) \end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Usually the steel reinforcement is disposed symmetrically about the axis of the arch; in such case the determination of the resultant axis is eliminated, since it coincides with the axis of the arch, and we may write in the above formulæ $e_1=e_2=0$.

2. Design by Phase II

Neglecting or Discounting the Effectiveness of the Concrete in Tension

Again, let

b = breadth of the arch-strip;
d = its depth of section;
A_s = cross-section of the steel reinforcement, and
j = distance of its center of gravity from the extreme compression fibers of the section;
x = distance of the neutral axis from the same fibers;
$n = E_s : E_{cp}$ and $m = E_{ct} : E_{cp}$;
I_s = moment of inertia of the total steel section about its center of gravity;
N = the normal or axial pressure acting on the section, and
$M = N \cdot p_n$ = the external moment referred to the axis of the arch.

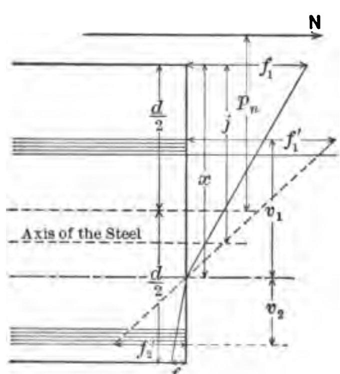

Fig. 5.—Reinforced Concrete Section. Phase II.

Let the stress in the concrete at unit distance from the neutral axis on the compression side of the section be denoted by f_u; then the stress at the same distance on the tension side of the concrete will be $m \cdot f_u$, and the stress in the steel will be $n \cdot f_u$.

The conditions of equilibrium between the internal and external forces yield the following two equations:

$$N = [\tfrac{1}{2}b \cdot x^2 - \tfrac{1}{2}m \cdot b(d-x)^2 + n \cdot A_s(x-j)] \cdot f_u, \quad . \quad . \quad (a)$$

$$N\left(p_n + x - \frac{d}{2}\right) = \{\tfrac{1}{3}bx^3 + \tfrac{1}{3}mb(d-x)^3 + n[I_s + A_s(x-j)^2]\} \cdot f_u, \quad (b)$$

from which, by division, we obtain the following cubic equation for determining x:

$$bx^3 + 3bx^2\left(p_n - \frac{d}{2}\right) - mb(d-x)^2\left(x + 3p_n + \frac{d}{2}\right) + 6n \cdot A_s(x-j)\left(p_n + j - \frac{d}{2}\right) - 6n \cdot I_s = 0. \quad (7)$$

With the value of x thus determined, we calculate by Eq. (a) or (b) the unit stress f_u; from this the critical normal stresses are obtained as follows:

in the concrete $\begin{cases} \text{stress in compression fiber, } f_1 = f_u \cdot x, \\ \text{stress in tension fiber, } -f_2 = m \cdot f_u(d-x), \end{cases}$

in the steel $\begin{cases} \text{stress in compression fiber, } f'_1 = n \cdot f_u \cdot v_1, \\ \text{stress in tension fiber, } -f'_2 = n \cdot f_u \cdot v_2. \end{cases}$

With $m = 0$, the above working formulæ pass over into Phase II.

By the above analysis, the normal fiber stresses at any section of an arch may be calculated if the effective external force, namely the normal component N of the joint-pressure, and its moment M about the axis of the arch are known. It is generally unnecessary to investigate the shearing stresses in an arch since these are always negligible except where large concentrated loads are brought upon the arch (through columns). For calculating the shears, the expressions for straight beams may be used with sufficient approximation.

CHAPTER III

THE THREE-HINGED ARCH

IT has been explained in Chapter I that all the forces acting on an arch are statically determinate if three points of the line of resistance are given. Such points may be fixed

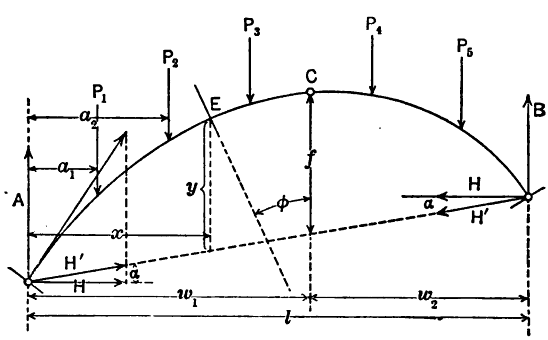

FIG. 6.—The Three-hinged Arch.

by restricting the pressure at three joints of the arch to very small areas or by introducing hinges at those points. We will assume complete freedom from friction at these hinges, and we may then quickly locate the end-reactions and line of resistance for any loading.

First let us consider the general case of a three-hinged arch subjected to a number of vertical loads P (Fig. 6). Let the horizontal distances of the crown-hinge from the end-hinges be w_1 and w_2, and its height above their connecting chord be f. Consider each of the two end-reactions resolved into

a vertical force, A or B, and a force in the direction of the closing chord equal to $H' = H \cdot \sec \alpha$. Then,

$$A = \frac{1}{l} \Sigma P(l-a), \quad B = \frac{1}{l} \Sigma Pa.$$

The forces A and B are thus identical with the reactions of a beam subjected to the same loading and having the same span l.

At any point E having an altitude y above the chord AB, the bending moment is given by

$$M = Ax - \sum_0^x P(x-a) - H \cdot y = M_b - H \cdot y. \quad . \quad . \quad . \quad (8)$$

Here M_b denotes the moment at the point E of a similarly loaded *beam*, and is calculated directly from the given loading.

At the hinges, neglecting all frictional resistance, the moment $M = 0$. Consequently, if we apply Eq. (8) to the crown-hinge C, and let M_c denote the moment of the vertical forces about the point C, we obtain, $0 = M_c - H \cdot f$; hence

$$H = \frac{M_c}{f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

There is thus determined the horizontal thrust of the three-hinged arch for any vertical loading; and, by Eq. (8), the bending moment at any point may then be found.

If a section is passed through the point E at an inclination ϕ to the vertical, the axial thrust will be

$$N = (A - \sum_0^x P) \sin \phi + H \cdot \tan \alpha \cdot \sin \phi + H \cdot \cos \phi$$
$$= (S_b + H \cdot \tan \alpha) \sin \phi + H \cdot \cos \phi. \quad (10)$$

Here S_b denotes the transverse shear at the point E of a corresponding simple beam.

If the external loading consists of inclined forces (e.g., wind-pressure on roof-arches), the graphic procedure becomes

most advantageous. For a single concentration (Fig. 7), the end-reaction may be obtained very simply by resolution of forces: one of the reaction-lines is given by the line connecting the hinges B and C; the direction of the other reaction is then also known; and the magnitudes of the two reactions are then determined by the triangle of forces.

If there are a number of loads, they are divided into two groups acting on the arch-segments AC and BC respectively. We find the resultant of each group and, by the above procedure, resolve it into its end-reactions. We thus obtain two

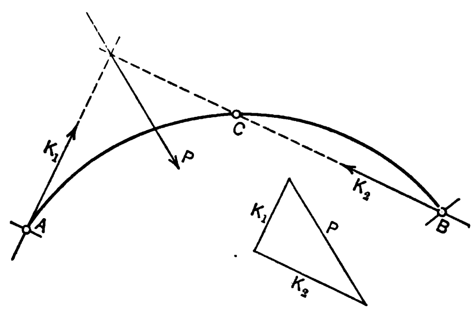

FIG. 7.—Reactions for a Single Concentration.

forces at each end, K'_1, K''_1, and K'_2, K''_2, whose composition gives the resultant end-reactions K_1 and K_2. This treatment is illustrated in Fig. 8. The funicular polygon of the forces P, constructed with the above end-reactions and the resulting pole O, finally gives the line of resistance passing through the given points A, C, B, or the hinges of the arch.*

* *Translator's Note:* To construct the line of resistance for a three-hinged arch, any of the common methods of passing an equilibrium polygon through three points may be used. The method recommended above and illustrated in Fig. 8 is as follows: Plot the forces acting on the two segments of the arch as a continuous polygon, and from the point of junction draw two rays parallel to the chords AC and BC respectively. For the group of forces acting on the segment AC, take a trial pole at any point O_1 on the ray $\parallel BC$; similarly, for the other group of forces, take a pole O_2 on the ray $\parallel AC$. The resulting trial equilibrium polygons are C–2–1–A' and C–3–4–B'. Then Z_1 and Z_2, the intersections of the terminal sides, are points of the respective resultants; and AZ_1 and BZ_2 are the directions of the end-reactions. The rays K'_1 and K''_2 are then drawn parallel

The procedure becomes simplified when the loading is symmetrical about a vertical line passing through the crown-hinge and bisecting the arch-chord, for the direction of the thrust at the crown is then known in advance, being parallel to the arch-chord. This case occurs in an arch of symmetrical form, symmetrically loaded. Thus Fig. 9 shows the construction of the line of resistance corresponding to the dead load of a symmetrical three-hinged arch. In order to replace the continuous loading by concentrations, the arch is divided

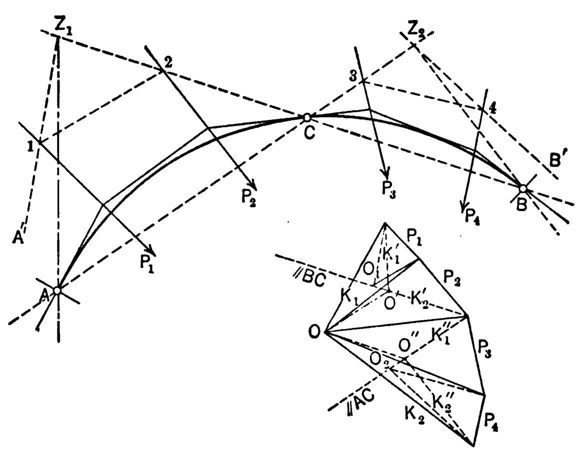

FIG. 8.—Construction of the Line of Resistance for a Three-hinged Arch.

into segments; the weight of each segment, together with that of the spandrel filling or other load which it supports, is assumed concentrated at the resultant center of gravity. In accordance with the remarks on this matter in Chapter I, the sections are taken in radial directions across the rib and then continued vertically upward through the spandrel filling or superimposed masonry. In bridge-arches, in which no great pitch of extrados occurs, only the vertical pressure of the spandrel filling need be considered; in other cases, as in tunnel-

to these end-reactions, giving the true poles, O' and O'', for the respective segments. Completing the parallelogram between these two poles, we obtain the resultant pole O for the true equilibrium polygon or line of resistance.

arches, the horizontal component or conjugate thrust of the filling would also require consideration. The calculated weights of the segments are drawn as force-vectors; the arch being symmetrical, it suffices to carry out the construction for but one-half of the span. In the funicular polygon of the loads, the side passing through the hinge will be parallel to the arch-chord, hence horizontal; this fixes the pole-line, $O'5'$, on which we take any point O' as a trial pole. The resulting funicular polygon will not pass through the hinge-point A, if O' has been arbitrarily chosen; but the last side will intersect the one through the crown in some point Z lying on the resultant of the applied loads; this constitutes a point through

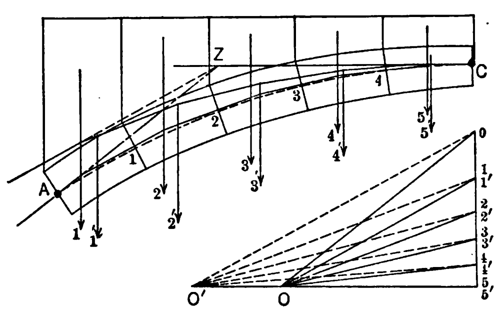

FIG. 9.—Line of Resistance for a Symmetrical Arch.

which the true end-reaction must pass. As soon as this true reaction-line ZA is thus found, a ray drawn parallel thereto in the force polygon gives the magnitude of the reaction and fixes the true pole O with which the correct resistance-polygon may be constructed. On account of the continuous distribution of the loading, this polygon should be replaced by a continuous curve passing through the points in which the sides of the polygon pierce the joints of the arch.

In comparison with the graphic procedure, the analytic determination of the joint-pressures possesses the advantage of greater precision; since, in the former, a small displacement of the line of resistance may produce considerable changes in

the distribution of stresses, especially if the radial depth is small.

If the loading is uniformly distributed (live load), the design is again very simple. For a uniform load of intensity p per unit length, if the crown-hinge is at the mid-point of the span, Eq. (9) gives:

$$H_p = \frac{1}{8}\frac{pl^2}{f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If H_g, the horizontal thrust due to dead load, has already been determined, we proceed as follows: For the completely loaded arch, $H = H_g + H_p$; drawing a new force polygon for the combined dead and live loads, and using a pole-distance $H_g + H_p$, the resulting funicular polygon will be the line of resistance for the completely loaded arch. This full loading, however, yields maximum stresses only for the sections near the crown- and end-hinges; at other points, a partial loading will be more severe in effect, and, as will be shown later, each section has its own critical (severest) loading. In smaller bridge arches, however, in addition to the fully loaded condition, it will suffice to investigate simply for a half-span load, since this produces the greatest displacement of the line of resistance from its mean position and, consequently, the greatest stresses near the quarter-points of the span. The line of resistance for the half-load may be constructed as follows (Fig. 10): To find the pole of the force polygon, augment the horizontal thrust H_g by $\frac{1}{2}H_p$ and then correct for the vertical end-reaction due to the live load. This amounts to $\frac{1}{8}pl$ at the end of the unloaded segment of the arch. Hence, if the right half of the span is loaded, the pole for constructing the line of resistance lies midway between the two poles for dead load and live load and at a distance $\frac{1}{8}pl$ below the line joining these poles.

In applying the analytic method, it is best not to combine the dead and live loads, but to calculate the stresses for each

separately, so that we may take advantage of the very simple formulæ for uniformly distributed loads:

For a full-span load, Eq. (8) gives the bending moment at any arch-point (x, y), as

$$M=\frac{1}{2}px(l-x)-\frac{1}{8}\frac{pl^2}{f}\cdot y. \quad . \quad . \quad . \quad . \quad (12)$$

If the arch-points lie on a parabola, $y=\frac{4f}{l^2}\cdot x\cdot(l-x)$, then $M=0$. Hence in the three-hinged arch of parabolic form,

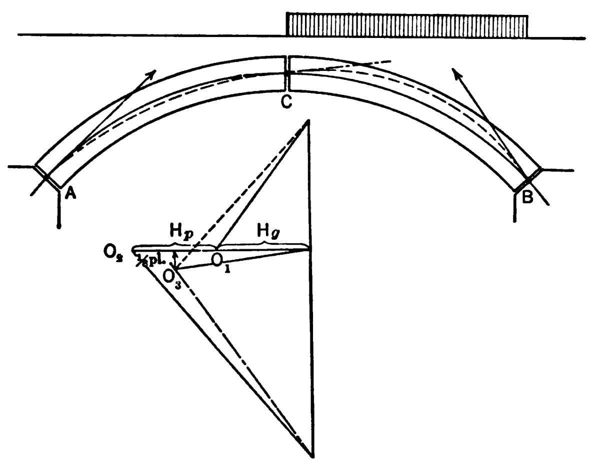

FIG. 10.—Line of Resistance for a Half-span Load.

uniformly loaded over the entire span, only axial stress occurs; i.e., the pressure is uniformly distributed over each section.

With one-half of the span loaded, the bending moments in the loaded half will be

$$M'=\frac{1}{8}px(3l-4x)-\frac{1}{16}\frac{pl^2}{f}\cdot y$$

and in the unloaded half,

$$M''=\frac{1}{8}plx-\frac{1}{16}\frac{pl^2}{f}\cdot y \quad . \quad . \quad . \quad . \quad (13)$$

The abscissa x, in Eq. (13), is measured in each case from the nearer end of the arch. The moments at the quarter-points of the arch are, accordingly,

$$M' = \frac{1}{16} pl^2 \left(1 - \frac{y}{f}\right) \quad \text{and} \quad M'' = -\frac{1}{16} pl^2 \left(\frac{y}{f} - \frac{1}{2}\right).$$

For a parabolic arch-axis these moments become,

$$M' = -M'' = \tfrac{1}{64} pl^2.$$

If the two resistance-lines, for loading on the left and right halves of the arch respectively, are considered as the limiting positions of the line of resistance, their mean curve will give the best form for the axis of the arch; because, for these extreme cases of loading, the two resistance-lines will then pass equally close to the upper and lower extreme fibers of each section, respectively, so that these fibers will receive equal or nearly equal stresses. But the ordinates of the lines of resistance for the half-span loadings are

$$\frac{1}{H_g + \frac{1}{2} H_p}(M' + M_g) \quad \text{and} \quad \frac{1}{H_g + \frac{1}{2} H_p}(M'' + M_g),$$

hence the mean ordinates are,

$$\frac{1}{H_g + \frac{1}{2} H_p}\left(\frac{M' + M''}{2} + M_g\right) = \frac{1}{H_g + \frac{1}{2} H_p}(\tfrac{1}{2} M_p + M_g);$$

i.e., the mean curve (or best arch-curve) coincides with the *line of resistance for the dead load and one-half the live load covering the entire span.* In a rational determination of the arch-curve, this line of resistance should be made the pattern for the axis of the arch.

The largest deviation of the line of resistance from the arch-axis will then amount to $\frac{1}{64} \frac{pl^2}{H_g + \frac{1}{2} H_p}$. By reducing the span l this deviation is diminished. It will therefore usually

be advantageous to locate the hinges some distance within the springing-points, and treat the resulting end-portions as parts of the abutments (Fig. 11).

The more exact investigation of large bridge-arches demands the consideration of the critical loading for each individual arch-section, and the problem of moving concentrations may also occur. In such case, the method of influence lines is to be applied.

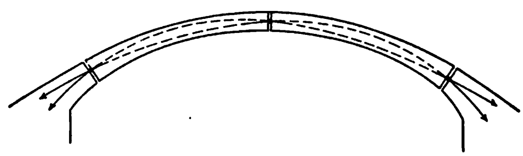

FIG. 11.—Location of the Hinges.

By Eq. (1), the general expression for the extreme fiber-stresses is

$$f_{1,2}=\frac{N}{A}\pm\frac{M\cdot a_{1,2}}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Here N is the axial thrust, M is the moment of the external forces referred to the arch-axis. Introducing the core-points of the cross-section, we may also write

$$f_1=\frac{M_1\cdot a_1}{I} \quad \text{and} \quad f_2=-\frac{M_2\cdot a_2}{I},$$

where M_1 and M_2 denote the moments about the core-points K_1 and K_2 respectively. The extreme fiber-stresses are thus proportional to the core-point moments, and the influence lines for the latter should therefore be determined. In order to save the determination of the two core-lines, and to save calculating the stresses at each section by two different influence lines and for two different loadings, we may employ the admissible simplification of determining the stresses by Eq. (1) and

using the same critical loading for both extreme fiber stresses at any section: namely, that loading which renders the moment M, about the center of the section, a maximum or minimum. It is therefore necessary to determine the influence lines for M and N.

By Eq. (8), $M=M_b-Hy=y\left(\frac{M_b}{y}-H\right)$. Consequently M may be represented by the difference between the influence quantities $\frac{M_b}{y}$ and H. But, by Eq. (9), the H influence line corresponds to that for the crown moments M_c, and is therefore a triangle, whose altitude, located at the crown-hinge, is $G\frac{w_1w_2}{f\cdot l}$, or, for a centrally located crown-hinge, $G\frac{l}{4f}$. Similarly, the influence line for $\frac{M_b}{y}$ is formed of two straight lines which intercept, on the left end-vertical, the distance $G\frac{x}{y}$. Retaining the H-curve unchanged, the influence quantities $\frac{M_b}{y}-H$ are readily constructed for all the arch-points. For the arch-point E, they are represented by the shaded ordinates in Fig. 12(a). These must be measured with a scale whose unit is the length assumed for G, and multiplied by the ordinate y and by the actual value of the load to give the moment M. All loads applied between A and J produce positive moments, those between J and B negative moments. The critical point J (i.e., the division point of the loading) may also be determined by the intersection of the reaction lines, AE and BC.

According to the well-known properties of influence lines, the effect of a uniformly distributed load is given by the area of the influence figure, under the given load, multiplied by the load per unit length and by the multiplier y; to find the effect of any number of concentrated loads, the influence ordinate under each load must be multiplied by the value of that load and by the ordinate y. It is thus a simple matter

to determine the severest effects of the loading, namely M_{max} and M_{min}, for each point of the arch.

For the same conditions of loading, the axial thrusts remain to be calculated. By Eq. (10),

$$N = S_b \cdot \sin\phi + H(\tan\alpha \cdot \sin\phi + \cos\phi),$$

or

$$N = \frac{\cos(\phi-\alpha)}{\cos\alpha}\left(S_b \cdot \frac{\sin\phi \cdot \cos\alpha}{\cos(\phi-\alpha)} + H\right).$$

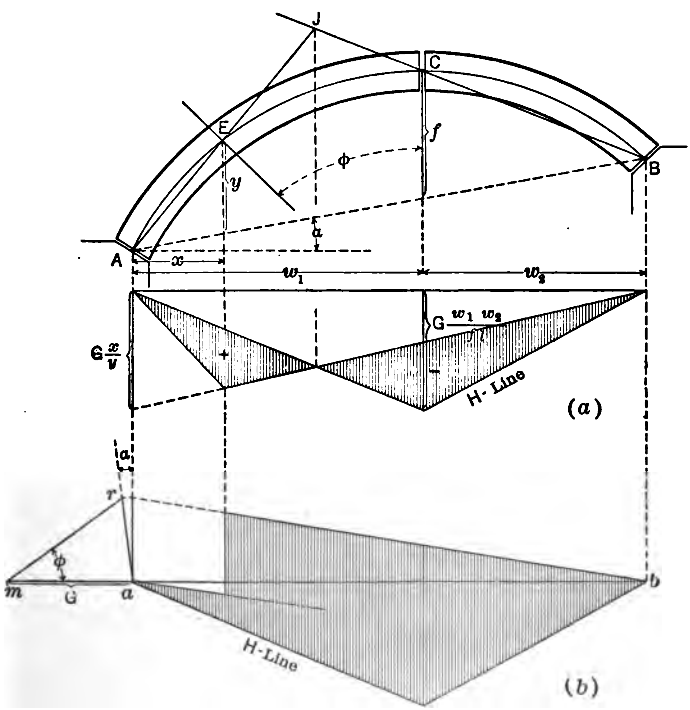

FIG. 12.—Influence Lines for the Three-hinged Arch.

The first part of the expression in brackets represents the shear producible in a simple beam by a load $G\,\frac{\sin\phi\cdot\cos\alpha}{\cos(\phi-\alpha)}$, and this may readily be constructed or calculated. Adding this to the H-curve, we obtain the influence line for N as shown in Fig.

12(b); the ordinates of the shaded area, measured to a scale whose unit is G, must be multiplied by $\frac{\cos(\phi-\alpha)}{\cos\alpha}$, or the ordinates may be measured directly by taking the length $G\frac{\cos\alpha}{\cos(\phi-\alpha)}=\overline{mr}$ as the scale-unit.

From the maximum moments M and the corresponding values of N, found as just described, we may calculate the maximum extreme fiber stresses by Eq. (1); the greatest compression in the upper fibers and tension in the lower fibers being given by the positive maximum of M, and the greatest compression in the lower fibers and tension in the upper fibers being given by the negative maximum of M.

To these stresses there should of course be added those due to the dead load.

CHAPTER IV

THE HINGELESS ARCH

In the vast majority of cases, the arches we have to deal with (in plain or reinforced concrete) are without any hinges, their ends being rigidly joined to the abutments. We thus have hingeless arches in which the external reactions must be determined from the conditions governing the elastic deformations. Usually we may consider the abutments as rigid and assume the ends of the arch at points where rotation of the section is practically impossible. In other cases, with elastic yielding abutments, the effect of imperfect constraint must be separately considered. (See Chapter IX.)

In calculating the elastic reactions of a concrete arch, we may apply the general principles of elastic arches; but for this we have to presuppose a material of uniform and constant elasticity and to know or assume the sectional constants (areas and moments of inertia). The difficulty may arise in arches in which tensile stresses occur that a part of the cross-section in the tension portions becomes ineffective and drops out. Such tensile stresses, in a properly curved arch, will first appear near the ends, reducing the effective cross-section there and partially releasing the constraint. Under such conditions the computation of the arch as constrained, with the full sections effective, can no longer yield entirely correct results. In a well-proportioned masonry arch, with the necessary increase in depth at the ends, the above condition will not arise, however, except as a result of unusual secondary influences such as temperature changes or yielding of abutments. In reinforced

concrete arches, on the other hand, tensile stresses are allowed under ordinary conditions, since the reinforcement takes care of them. The constraint therefore is not lost; but we encounter the difficulty, particularly in arches in which the axial thrust is small compared with the bending moments, that a part of the arch is in a condition which renders inaccurate the assumption that the entire cross-section of the concrete is effective (as assumed in Phase I). Nevertheless, the error in the deformations calculated by Phase I will never be large, since these deformations represent the combined effect of the stresses in all the sections, whereas the deviations from the conditions of Phase I are always limited to small isolated portions of the span and therefore cannot appreciably affect the total result.

Furthermore we are compelled to use Phase I in calculating the deformations in all other concrete-steel structures, even in those suffering greater deflections than arches, since any other method of calculation would lead to practically insurmountable difficulties.

We therefore calculate the external reactions of a statically indeterminate arch of masonry or reinforced concrete by the formulæ applicable to an elastic arch of homogeneous material; substituting, in the case of a reinforced concrete arch, the values of the equivalent section A and moment of inertia I as given by Eqs. (3) and (4).

In arches with fixed ends, as shown in Chapter I, there are lacking in general three conditions for the static determination of the external reactions. These may be derived from the *three* conditions of constraint to which the elastic deformations of the system are subject. For this purpose we may employ either an analytic or a graphic treatment.

a. Analytic Method

Consider the arch (Fig. 13) cut in two by a vertical section at the crown C, and treat the joint-pressure at that section as an external load. This may be resolved into vertical and

horizontal components, S and H, acting at the center of gravity of the cross-section, and a moment M_0 about the same point; these forces are of course assumed as acting in contrary directions on the right and left halves of the arch. The total deformations for each half of the arch are obtained by summing the deformations of all the arch elements; but in this calculation we are compelled to keep the elements of finite length. We therefore introduce a number of sections o to n, or o to n', at

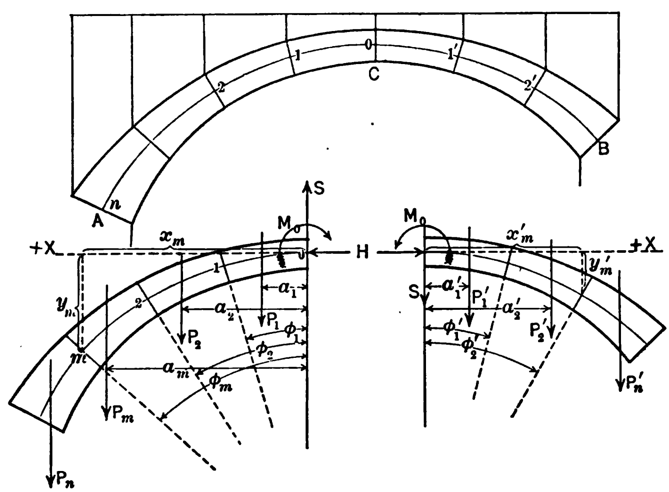

FIG. 13.—Analytic Method for the Hingeless Arch.

uniform distances Δs along the arch-axis, and calculate the weight of each division with its superimposed load. Let these weights be denoted by $P_1, P_2, \ldots P_n$ in the left half, and by $P'_1, P'_2, \ldots P'_n$ in the right half of the arch. The lines of action of the loads P are specified by the abscissæ a; and the points of the arch-axis are specified by their coordinates (x, y) measured from a pair of rectangular axes drawn through the arch-point o.

For the moment M and the axial thrust N at the various sections, we obtain the following expressions:

Left side of the arch:

At section 0,
$$M_0 = M_0.$$
At section 1,
$$M_1 = M_0 + P_1(x_1 - a_1) - H \cdot y_1 - S \cdot x_1.$$
. . . .

At section m,
$$M_m = M_0 + \sum_{r=1}^{r=m} P_r(x_m - a_r) - H \cdot y_m - S \cdot x_m.$$
. . . .

At section 0,
$$N_0 = H.$$
At section 1,
$$N_1 = H \cdot \cos \phi_1 + P_1 \cdot \sin \phi_1 - S \cdot \sin \phi_1.$$
. . . .

At section m,
$$N_m = H \cdot \cos \phi_m + (\sum_1^m P) \sin \phi_m - S \cdot \sin \phi_m.$$
. . . .

Right side of the arch: . (14)

At section 0,
$$M'_0 = M_0.$$
At section 1,
$$M'_1 = M_0 + P'_1(x'_1 - a'_1) - H \cdot y'_1 + S \cdot x'_1.$$
. . . .

At section m,
$$M'_m = M_0 + \sum_{r=1}^{r=m} P'_r(x'_m - a'_r) - H \cdot y'_m + S \cdot x'_m.$$
. . . .

At section 0,
$$N'_0 = H.$$
At section 1,
$$N'_1 = H \cdot \cos \phi'_1 + P'_1 \cdot \sin \phi'_1 + S \cdot \sin \phi'_1.$$
. . . .

At section m,
$$N'_m = H \cdot \cos \phi'_m + (\sum_1^m P') \sin \phi'_m + S \cdot \sin \phi'_m.$$
. . . .

In the above expressions, the quantities H, S and M_0 are as yet unknown. To determine them, we will apply the conditions governing the deformations of the system. Let W_1 denote the elastic work of the deforming forces on the left side of the arch, and W_2 the corresponding quantity for the right side of the arch. Then, if the end-sections are absolutely incapable of displacement or rotation, the partial differential coefficient of W_1 and W_2 with respect to H, S and M_0 (by Castigliano's Theorem), will give the displacements and rotation of the section o in the direction of H, S and M_0; these must be equal for the two halves of the arch, but opposite in sign, so that the corresponding sums must be zero. Consequently, if $W = W_1 + W_2$ is the total work of deformation, we have the condition:

$$\frac{\partial W}{\partial H} = 0, \qquad \frac{\partial W}{\partial S} = 0, \qquad \frac{\partial W}{\partial M_0} = 0.$$

For straight and approximately for curved ribs, neglecting the deflections due to shearing stresses, we have

$$W = \frac{1}{2}\int_A^B \frac{M^2}{EI} ds + \frac{1}{2}\int_A^B \frac{N^2}{EA} ds.$$

Hence, assuming a constant elastic modulus and replacing the definite integrals by summations, the three equations of condition become:

$$\left.\begin{aligned}
&\Sigma \frac{M}{I} \cdot \frac{\partial M}{\partial H} \cdot \Delta s + \Sigma \frac{N}{A} \cdot \frac{\partial N}{\partial H} \cdot \Delta s = 0 \\
&\Sigma \frac{M}{I} \cdot \frac{\partial M}{\partial S} \cdot \Delta s + \Sigma \frac{N}{A} \cdot \frac{\partial N}{\partial S} \cdot \Delta s = 0 \\
&\Sigma \frac{M}{I} \cdot \frac{\partial M}{\partial M_0} \cdot \Delta s = 0.
\end{aligned}\right\} \quad . \quad . \quad . \quad (15)$$

Substituting in Eqs. (15) the expressions for M and N and their differential coefficients from Eqs. (14), and introducing the following abbreviations:

$$\begin{cases} \sum\limits_{r=1}^{r=m} P_r(x_m-a_r) = \mathfrak{M}, \\ \sum\limits_{r=1}^{r=m} P'_r(x'_m-a'_r) = \mathfrak{M}', \end{cases} \qquad \begin{cases} \sum\limits_{1}^{m} P = \mathfrak{P}, \\ \sum\limits_{1}^{m} P' = \mathfrak{P}', \end{cases}$$

we obtain the following three linear equations for determining H, S and M_0:

$$\left.\begin{array}{l}
\left.\begin{array}{l}
-M_0\sum\limits_A^B \frac{y}{I}\Delta s + H\left[\sum\limits_A^B \frac{y^2}{I}\Delta s + \sum\limits_A^B \frac{\cos^2\phi}{A}\Delta s\right] \\
\quad + S\left[\sum\limits_A^C \frac{xy}{I}\Delta s - \sum\limits_C^B \frac{xy}{I}\Delta s - \sum\limits_A^C \frac{\sin\phi\cdot\cos\phi}{A}\Delta s \right. \\
\quad \left. + \sum\limits_C^B \frac{\sin\phi\cdot\cos\phi}{A}\Delta s\right] - \sum\limits_A^B \frac{\mathfrak{M}y}{I}\Delta s \\
\quad + \sum\limits_A^B \mathfrak{P}\frac{\sin\phi\cdot\cos\phi}{A}\Delta s = 0.
\end{array}\right\} (a) \\
\left.\begin{array}{l}
-M_0\left[\sum\limits_A^C \frac{x}{I}\Delta s - \sum\limits_C^B \frac{x}{I}\Delta s\right] + H\left[\sum\limits_A^C \frac{xy}{I}\Delta s - \sum\limits_C^B \frac{xy}{I}\Delta s \right. \\
\quad \left. - \sum\limits_A^C \frac{\sin\phi\cdot\cos\phi}{A}\Delta s + \sum\limits_C^B \frac{\sin\phi\cdot\cos\phi}{A}\Delta s\right] \\
\quad + S\left[\sum\limits_A^B \frac{x^2}{I}\Delta s + \sum\limits_A^B \frac{\sin^2\phi}{A}\Delta s\right] - \sum\limits_A^C \frac{\mathfrak{M}x}{I}\Delta s \\
\quad + \sum\limits_C^B \frac{\mathfrak{M}'x}{I}\Delta s - \sum\limits_A^C \mathfrak{P}\frac{\sin^2\phi}{A}\Delta s \\
\quad + \sum\limits_C^B \mathfrak{P}'\frac{\sin^2\phi}{A}\Delta s = 0.
\end{array}\right\} (b) \\
\left.\begin{array}{l}
M_0\sum\limits_A^B \frac{\Delta s}{I} - H\sum\limits_A^B \frac{y}{I}\Delta s - S\left[\sum\limits_A^C \frac{x}{I}\Delta s - \sum\limits_C^B \frac{x}{I}\Delta s\right] \\
\quad + \sum\limits_A^B \frac{\mathfrak{M}}{I}\Delta s = 0
\end{array}\right\} (c)
\end{array}\right\} . \quad (16)$$

or, in abbreviated form,

$$\left.\begin{array}{ll}\alpha M_0+\beta H+\gamma S+A_1=0 & (a)\\ \delta M_0+\gamma H+\epsilon S+A_2=0 & (b)\\ \phi M_0+\alpha H+\delta S+A_3=0 & (c)\end{array}\right\} \quad . \quad . \quad . \quad . \quad (17)$$

The coefficients α, β, γ, δ, ϵ, ϕ, depend only on the form of the arch and the sectional quantities A and I; whereas the quantities A_1, A_2, A_3, depend also upon the loading. The summations entering these expressions are to be calculated from the individual values for the sections o–n and o–n', by Simpson's rule of summation.

If the arch is symmetrical about the section C, the coefficients γ and δ both vanish and the above equations reduce to the following:

$$\left.\begin{array}{ll}\alpha M_0+\beta H+A_1=0 & (a)\\ \phi M_0+\alpha H+A_3=0 & (c)\\ \epsilon S+A_2\ =0 & (b)\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Here, assuming n to be an even number and taking the factor $\frac{2}{3}\cdot\Delta s$ out of all the summations, the coefficients represent the following expressions:

$$\left.\begin{array}{l}\alpha=-\left[\frac{4y_1}{I_1}+\frac{2y_2}{I_2}+\frac{4y_3}{I_3}+\ldots\frac{y_n}{I_n}\right]\\ \beta=\left[4\frac{y_1^2}{I_1}+2\frac{y_2^2}{I_2}+4\frac{y_3^2}{I_3}+\ldots\frac{y_n^2}{I_n}\right]\\ \quad+\left[\frac{1}{A_0}+4\frac{\cos^2\phi_1}{A_1}+2\frac{\cos^2\phi_2}{A_2}+4\frac{\cos^2\phi_3}{A_3}+\ldots\frac{\cos^2\phi_n}{A_n}\right]\\ \phi=\left[\frac{1}{I_0}+\frac{4}{I_1}+\frac{2}{I_2}+\frac{4}{I_3}+\ldots\frac{1}{I_n}\right]\\ \epsilon=\left[4\frac{x_1^2}{I_1}+2\frac{x_2^2}{I_2}+4\frac{x_3^2}{I_3}+\ldots\frac{x_n^2}{I_n}\right]\\ \quad+\left[4\frac{\sin^2\phi_1}{A_1}+2\frac{\sin^2\phi_2}{A_2}+4\frac{\sin^2\phi_3}{A_3}+\ldots\frac{\sin^2\phi_n}{A_n}\right]\end{array}\right\} \quad (19)$$

$$
\left.\begin{aligned}
A_1 &= -\frac{1}{2}\left[4\frac{y_1}{I_1}(\mathfrak{M}_1+\mathfrak{M}'_1)+2\frac{y_2}{I_2}(\mathfrak{M}_2+\mathfrak{M}'_2)+4\frac{y_3}{I_3}(\mathfrak{M}_3+\mathfrak{M}'_3)\right.\\
&\quad\left.+\ldots\frac{y_n}{I_n}(\mathfrak{M}_n+\mathfrak{M}'_n)\right]+\frac{1}{2}\left[4\frac{\cos\phi_1\cdot\sin\phi_1}{A_1}(\mathfrak{P}_1+\mathfrak{P}'_1)\right.\\
&\quad+2\frac{\cos\phi_2\cdot\sin\phi_2}{A_2}(\mathfrak{P}_2+\mathfrak{P}'_2)+4\frac{\cos\phi_3\cdot\sin\phi_3}{A_3}(\mathfrak{P}_3+\mathfrak{P}'_3)\\
&\quad\left.+\ldots\frac{\cos\phi_n\cdot\sin\phi_n}{A_n}\cdot(\mathfrak{P}_n+\mathfrak{P}'_n)\right].\\
A_2 &= -\frac{1}{2}\left[4\frac{x_1}{I_1}(\mathfrak{M}_1-\mathfrak{M}'_1)+2\frac{x_2}{I_2}(\mathfrak{M}_2-\mathfrak{M}'_2)+4\frac{x_3}{I_3}(\mathfrak{M}_3-\mathfrak{M}'_3)\right.\\
&\quad\left.+\ldots\frac{x_n}{I_n}(\mathfrak{M}_n-\mathfrak{M}'_n)\right]-\frac{1}{2}\left[4\frac{\sin^2\phi_1}{A_1}(\mathfrak{P}_1-\mathfrak{P}'_1)\right.\\
&\quad+2\frac{\sin^2\phi_2}{A_2}(\mathfrak{P}_2-\mathfrak{P}'_2)+4\frac{\sin^2\phi_3}{A_3}(\mathfrak{P}_3-\mathfrak{P}'_3)\\
&\quad\left.+\ldots\frac{\sin^2\phi_n}{A_n}(\mathfrak{P}_n-\mathfrak{P}'_n)\right].\\
A_3 &= \frac{1}{2}\left[\frac{4}{I_1}(\mathfrak{M}_1+\mathfrak{M}'_1)+\frac{2}{I_2}(\mathfrak{M}_2+\mathfrak{M}'_2)+\frac{4}{I_3}(\mathfrak{M}_3+\mathfrak{M}'_3)\right.\\
&\quad\left.+\ldots\frac{1}{I_n}(\mathfrak{M}_n+\mathfrak{M}'_n)\right].
\end{aligned}\right\} \quad (20)
$$

Solving Eqs. (18):

$$
\left.H=\frac{\alpha A_3-\phi A_1}{\beta\phi-\alpha^2},\quad S=-\frac{A_2}{\epsilon},\quad M_0=\frac{\alpha A_1-\beta A_3}{\beta\phi-\alpha^2},\right\}\quad . \quad . \quad (21)
$$

If the loading, also, is symmetrical about the crown, then for similarly located points $\mathfrak{M}=\mathfrak{M}'$ and $\mathfrak{P}=\mathfrak{P}'$; hence $A_2=0$ and consequently the vertical crown-shear $S=0$. The crown thrust is then horizontal and is fixed by H and M_0.

If the live load is uniformly distributed, and if only two cases of loading are to be investigated, namely, full and half-load, it is necessary in the latter case merely to find S from Eq. (21); for H and M_0 are just one-half the corresponding quantities for full load.

With the values thus calculated for H, M_0, and S, the effective forces M and N at every section may be obtained by Eqs. (14). The internal stresses may then be calculated by the rules given in Chapter II.*

For a more precise calculation of the maximum stresses, a different condition of loading must be considered for each section. For this purpose, the influence values for a concentration applied at the successive points of the arch must be calculated and used in determining the maximum stresses by the method of influence lines. The above analytical method, however, then becomes very tedious, since for each position of the load we must calculate the values of $\mathfrak{M}$ and $\mathfrak{M}'$, then A_1, A_2, A_3, and then, with the results, solve the Eqs. (18). Hence the above method is practically useful only when very few cases of loading need to be investigated, as when the stresses are to be found simply for dead load and for live load covering the full or half-span.

The line of resistance is fixed by the values of M and N calculated by Eqs. (14); thus at the crown, $e_0 = \frac{M_0}{H}$, and at any section, $e_m = \frac{M_m}{N_m}$, give the radial distances of the line of resistance from the arch-axis (+ downward).

b. Graphic Method

Instead of starting with the work of elastic deformation, the equations giving the deflections of a curved rib may be derived in the following manner:

Let the arch AB (Fig. 14) be fixed at the end A but perfectly free at the end B. Consider two infinitely close sections, C, C', of the otherwise rigid rib to undergo a relative angular displacement of $d\psi$, but without changing the distance between their centers of gravity. As a result, if the portion AC remains fixed, the portion CB will swing through the angle $d\psi$, carrying

* It should be noted that the sign of M is here assumed opposite to that in the preceding and following chapters. (—Melan.)

the end B to a new position B', where BB' is perpendicular to CB. Let us refer the arch-point C by the coordinates (x, y) to the point B as origin. The vertical and horizontal displacements of the point B, on account of the similarity of the triangles $\Delta BB'B''$ and ΔCBD, and the relation $\overline{BB'}=\overline{CB}\cdot d\psi$, are given by

$$B'B''=x\cdot d\psi \quad \text{and} \quad BB''=y\cdot d\psi.$$

If, in addition to the relative rotation of the sections, C, C', there also occurs a change in the distance ds between their centers of gravity, the arch-element ds being shortened in length by Δds, then the point B will undergo an equal displacement

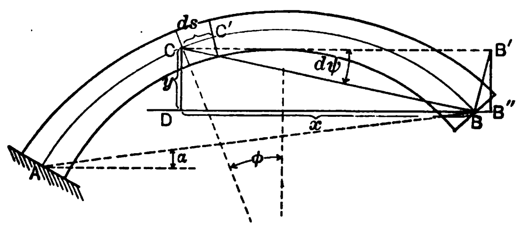

FIG. 14.—Deformations of an Arched Rib.

parallel to the element CC'; the components of this displacement will be $\Delta ds \cdot \sin \phi$ and $\Delta ds \cdot \cos \phi$. The total displacement of B is therefore given by

$$\begin{cases} B'B''=x\cdot d\psi-\Delta ds\cdot \sin \phi, \\ BB'' \;=y\cdot d\psi-\Delta ds\cdot \cos \phi. \end{cases}$$

The deformation of the arch-element CC' is produced by the moment M and axial thrust N acting on the section C. Hence by the theory of flexure of elastic bodies, if r is the radius of curvature, the following relations exist between the deformations of the elements and the applied forces:

$$\begin{cases} \dfrac{d\psi}{ds}=\dfrac{M}{EI}+\dfrac{N}{EAr}, \\ \dfrac{\Delta ds}{ds}=\dfrac{N}{EA}. \end{cases}$$

With these expressions we may obtain the elastic deformation of the arch-element, ds, and hence the resulting deflections of the point B, produced by the combination of a moment and an axial thrust acting on any section of the arch. The effect of the shearing forces is here neglected. If there are similar forces acting on all the other sections, so that all the arch-elements undergo deformation, the resulting total displacement of the end B may be obtained by summation of the individual elementary displacements. But, since the end B is here assumed absolutely immovable, its deflections must be equal to zero; we thus obtain the two equations of condition:

$$\int_A^B \frac{Mx}{EI}\cdot ds+\int_A^B \frac{N}{EA}\left(\frac{x}{r}-\sin\phi\right)ds=0. \quad . \quad . \quad . \quad (22)$$

$$\int_A^B \frac{My}{EI}\cdot ds+\int_A^B \frac{N}{EA}\left(\frac{y}{r}-\cos\phi\right)ds=0. \quad . \quad . \quad . \quad (23)$$

The fixedness of the end also requires that its rotational displacement should be zero; hence, since the end-section B is turned through the angle $d\psi$ with the rotation of each intermediate section, we must have $\int_A^B d\psi=0$; that is

$$\int \frac{M}{EI}\cdot ds+\int_A^B \frac{N}{EAr}\cdot ds=0. \quad . \quad . \quad . \quad . \quad (24)$$

The above three equations have been derived for a system of coordinate axes passing through B. They hold good, however, for any parallel transformation of the axes; for, if we substitute $x=x'+a$ and $y=y'+b$ in the first two equations, the added terms will vanish because of the condition imposed by the third equation. We will therefore move the origin of coordinates to some point O arbitrarily chosen for the present and assume the axis of abscissæ not horizontal but inclined at some angle τ. (Fig. 15). The points of the arch-axis are referred to O as origin, the abscissæ x being measured *horizontally*,

and the ordinates y being measured *vertically* from the inclined axis of abscissæ. Introducing these new coordinates, we must substitute $y-x\tan\tau$ for y in Eq. (23); hence, with the aid of Eq. (22) we obtain:

$$\left.\begin{aligned}&\int_A^B \frac{My}{EI}\cdot ds+\int_A^B \frac{N}{EA}\left(\frac{y}{r}-\frac{\cos(\phi-\tau)}{\cos\tau}\right)ds=0\\&\int_A^B \frac{Mx}{EI}\cdot ds+\int_A^B \frac{N}{EA}\left(\frac{x}{r}-\sin\phi\right)ds=0\\&\int_A^B \frac{M}{EI}\cdot ds+\int_A^B \frac{N}{EAr}ds=0\end{aligned}\right\}\quad . \quad . \quad (25)$$

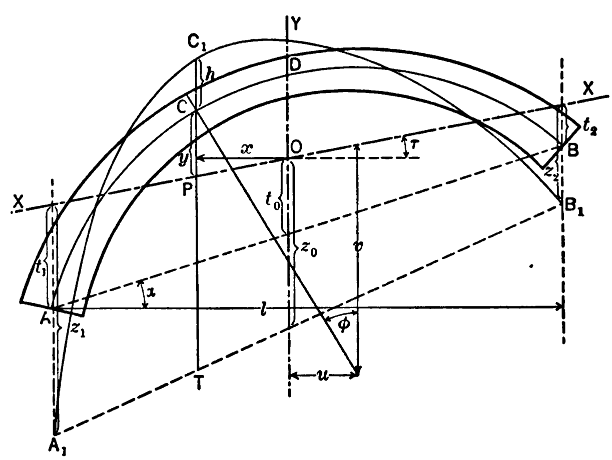

FIG. 15.—Line of Resistance for the Hingeless Arch.

The integrals containing N in the above expressions are always very small in value in comparison with the terms involving the bending moments. We may therefore safely introduce approximate expressions for the former terms. If u and v are the coordinates of the center of curvature of the arch-axis at the point C,

$$\frac{y}{r}-\frac{\cos(\phi-\tau)}{\cos\tau}=-\frac{v}{r},\quad\text{and}\quad\frac{x}{r}-\sin\phi=\frac{u}{r}.$$

If the arch-axis is assumed to be a "basket-handle" arch, i.e., composed of circular segments AB_1, B_1B_2, B_2B_3, . . . , then

$$\int_A^B \frac{N}{A}\left(\frac{y}{r}-\frac{\cos(\phi-\tau)}{\cos\tau}\right)ds=-\frac{v_1}{r_1}\int_A^{B_1}\frac{N}{A}\cdot ds-\frac{v_2}{r_2}\int_{B_1}^{B_2}\frac{N}{A}\cdot ds-\ldots$$

$$\int_A^B \frac{N}{A}\left(\frac{x}{r}-\sin\tau\right)ds=\frac{u_1}{r_1}\int_A^{B_1}\frac{N}{A}\cdot ds+\frac{u_2}{r_2}\int_{B_1}^{B_2}\frac{N}{A}\cdot ds+\ldots$$

If the asymmetry of the arch is not too great, we may, without sensible error, introduce here a mean value for v and for r and write the mean value of $u=0$. Also taking out the coefficient E, which is assumed constant, the Eqs. (25) become

$$\left.\begin{aligned}&\int_A^B \frac{My}{I}\cdot ds-\frac{v}{r}\int_A^B\frac{N}{A}\cdot ds=0\\&\int_A^B\frac{Mx}{I}\cdot ds=0\\&\int_A^B\frac{M}{I}\cdot ds+\frac{1}{r}\int_A^B\frac{N}{A}\cdot ds=0\end{aligned}\right\} \quad . \; . \; . \; . \quad (26)$$

On any section through C, inclined at an angle ϕ to the vertical, the axial thrust is

$$N=H\cos\phi+V\sin\phi,$$

where V is the vertical component of the pressure at the section. For the latter we may write $V=V_0+S_x$, V_0 denoting the vertical shear at the section D $(x=0)$ and S_x denoting the summation of the loading between D and C. Hence

$$\int_A^B\frac{N}{A}\cdot ds=H\int_A^B\frac{\cos\phi}{A}\cdot ds+V_0\int_A^B\frac{\sin\phi}{A}\cdot ds+\int_A^B\frac{S_x\cdot\sin\phi}{A}\cdot ds.$$

In any arch symmetrical about the axis of ordinates,

$$\int_A^B\frac{\sin\phi}{A}\cdot ds=0;$$

in general, however, since the arches used in practice are limited to very slight departures from perfect symmetry, this definite integral will be very small, so that the second term of the above expression may be neglected. The same is true of the third term in the case of symmetrical loading; and in all other cases this term may be represented by an insignificant correction quantity usually lying outside the limit of precision of the computations. Furthermore the latter quantity can be easily calculated; let it be denoted by q'. Noting that $ds \cdot \cos\phi = dx$ we obtain the expression

$$\int_A^B \frac{N}{A} \cdot ds = H \cdot \int_A^B \frac{dx}{A} \cdot + q'. \quad . \quad . \quad . \quad . \quad (27)$$

Let $A_1C_1B_1$ (Fig. 15) represent the line of resistance for any given loading, and h its vertical distance above the arch-axis at the point C; then the bending moment at this point will be

$$M = H \cdot h = H(\overline{C_1T} - \overline{PT} - y) = M_b - H(\overline{PT} + y),$$

where M_b again represents the bending moment in a simple beam of span l. Furthermore, with the notation of Fig. 15,

$$\overline{PT} = z_0 + \frac{z_1 - z_2}{l} \cdot x.$$

Consequently

$$M = M_b - Hy - H \cdot z_0 - H \cdot \frac{z_1 - z_2}{l} \cdot x.$$

For our unknown quantities let us choose the forces:

$$\left.\begin{array}{l} H \quad \text{and} \quad X_1 = H \cdot \dfrac{z_1 - z_2}{l} \\ \\ X_2 = H \cdot z_0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and the moment:

We thus have

$$M = M_b - H \cdot y - X_1 \cdot x - X_2. \quad . \quad . \quad . \quad . \quad (29)$$

Substituting this expression with Eq. (27) in Eqs. (26,) we obtain:

$$\int_A^B \frac{M_b \cdot y}{I} \cdot ds - H \int_A^B \frac{y^2}{I} \cdot ds - X_1 \int_A^B \frac{xy}{I} \cdot ds - X_2 \int_A^B \frac{y}{I} \cdot ds$$
$$- \frac{v}{r}\left(H \int_A^B \frac{dx}{A} + q'\right) = 0.$$

$$\int_A^B \frac{M_b \cdot x}{I} \cdot ds - H \int_A^B \frac{xy}{I} \cdot ds - X_1 \int_A^B \frac{x^2}{I} \cdot ds - X_2 \int_A^B \frac{x}{I} \cdot ds = 0$$

$$\int_A^B \frac{M_b}{I} \cdot ds - H \int_A^B \frac{y}{I} \cdot ds - X_1 \int_A^B \frac{x}{I} \cdot ds - X_2 \int_A^B \frac{1}{I} \cdot ds$$
$$+ \frac{1}{r}\left(H \int_A^B \frac{dx}{A} + q'\right) = 0.$$

In the above system of equations the position of the origin of coordinates O and the direction of the axis of abscissæ XX are arbitrary. They will now be so chosen as to simplify the solution of the equations as much as possible. This is accomplished by making

$$\int_A^B \frac{x}{I} \cdot ds = 0, \quad \int_A^B \frac{y}{I} \cdot ds = 0, \quad \int_A^B \frac{xy}{I} \cdot ds = 0, \quad . \quad (30)$$

for then we obtain

$$\left.\begin{aligned} H &= \frac{\int_A^B \frac{M_b y}{I} \cdot ds - \frac{v}{r} \cdot q'}{\int_A^B \frac{y^2}{I} \cdot ds + \frac{v}{r} \int_A^B \frac{dx}{A}} \\ X_1 &= \frac{\int_A^B \frac{M_b x}{I} \cdot ds}{\int_A^B \frac{x^2}{I} \cdot ds} \\ X_2 &= \frac{\int_A^B \frac{M_b}{I} \cdot ds + \frac{1}{r}\left(H \int_A^B \frac{dx}{A} + q'\right)}{\int_A^B \frac{ds}{I}} \end{aligned}\right\} \quad . \quad . \quad . \quad (31)$$

The Eqs. (30) fix the position of the origin of coordinates O and the direction of the axis of abscissæ XX. If we imagine all the elements of the arch loaded with the reciprocal moments of inertia $\frac{1}{I}$, the first two of Eqs. (30) will express the condition that *the origin of coordinates must be placed at the center of gravity of these weights;* while the third equation is satisfied when *the centrifugal moment of these weights referred to the two axes is zero.* As is known, the latter is the case when the two

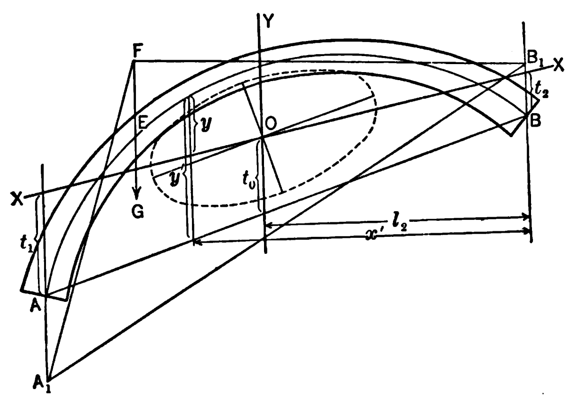

FIG. 16.—Graphic Method for Locating the Axes.

axes are conjugate with reference to the *inertia ellipse* of the weights $\frac{ds}{I}$. We may therefore locate the pair of axes by determining the center of gravity of all the arch elements loaded with $\frac{ds}{I}$, constructing the inertia ellipse (here called the *ellipse of elasticity*) and finding its axis conjugate to the vertical (Fig. 16). In actual application, however, the following analytical method of locating the pair of axes will be found preferable.

Referring to the right end of the arch B and to the closing chord AB, let x', y', be the coordinates of the points of the

arch, and l_2 and t_0 the coordinates of the desired point O. Let t_1 and t_2 be the distances intercepted on the end verticals by the X-axis passing through O. Then

$$x = x' - l_2,$$

$$y = y' - t_0 - \frac{t_1 - t_2}{l}(x' - l_2).$$

Substituting in Eqs. (30), we obtain

$$\left.\begin{aligned} l_2 &= \frac{\int_A^B \frac{x'}{I} \cdot ds}{\int_A^B \frac{ds}{I}} \\ t_0 &= \frac{\int_A^B \frac{y'}{I} \cdot ds}{\int_A^B \frac{ds}{I}} \\ \int_A^B \frac{xy'}{I} \cdot ds &- \frac{t_1 - t_2}{l} \int_A^B \frac{x^2}{I} \cdot ds = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (32)$$

The last equation determines $\frac{t_1 - t_2}{l}$ and thereby fixes the inclination of the axis of abscissæ to the chord of the arch.

As a rule, arches are built symmetrical about the vertical center line, so that any two points of the axis at equal distances $+x$ and $-x$ from the center line have equal values of y', also equal cross-sections. The resultant of the weights $\frac{ds}{I}$ then falls in the center line, which thus becomes the axis of ordinates; and since $\int \frac{xy'}{I} ds = 0$, $\frac{t_1 - t_2}{l} = 0$ also, i.e., *the axis of abscissæ is to be taken parallel to the closing chord of the arch.* But this

applies only under the assumption of the above described skew symmetry and, *a fortiori*, to arches squarely symmetrical about the center line with their ends at the same elevation.

In Eqs. (31), in comparison with the terms involving the moment of inertia I, those containing A are always very small, particularly those occurring in the numerators of H and X_2; if the radius of curvature r is not too small, the latter terms will always lie beyond the limits of precision of the computations and may therefore be omitted from the expressions. Furthermore, if the cross-section of the arch is assumed constant within the small horizontal lengths a_1, a_2 . . . and equal to A_1, A_2 . . . respectively, then

$$\int_A^B \frac{dx}{A} = \frac{a_1}{A_1} + \frac{a_2}{A_2} + \ . \ . \ . = \frac{l}{A_0},$$

where A_0 represents a mean area of cross-section. Substituting also

$$\left.\begin{aligned} \frac{y}{I}\cdot\frac{ds}{dx} &= \frac{y}{I\cdot\cos\phi} = w \\ \frac{x}{I}\cdot\frac{ds}{dx} &= \frac{x}{I\cdot\cos\phi} = w' \\ \frac{1}{I}\cdot\frac{ds}{dx} &= \frac{1}{I\cdot\cos\phi} = w'' \end{aligned}\right\} \quad . \ . \ . \ . \ . \qquad (33)$$

the Eqs. (31) may now be written:

$$H = \frac{\int_A^B M_b\, w\, dx}{\int_A^B y\, w\, dx + \frac{v}{r}\cdot\frac{l}{A_0}}, \quad X_1 = \frac{\int_A^B M_b\, w' dx}{\int_A^B x\, w' dx}, \quad X_2 = \frac{\int_A^B M_b\, w'' dx}{\int_A^B w'' dx}. \qquad (34)$$

We now assume the loading to consist of a single concentration G applied at the point E (Fig. 16).

The moments M_b in a simple beam similarly loaded are then represented by the ordinates of a triangle A_1FB_1; and this, as is well known, is identical with the influence line for moments at E produced by a load G moving over the span. The term $\int_A^B M_b w\, dx$ may therefore be conceived as the moment at the point E in a beam of span l loaded continuously with the quantities w. The same reasoning applies to $\int_A^B M_b w' dx$ and $\int_A^B M_b w'' dx$, if w' and w'' are considered the continuous loading. Similarly the integrals $\int_A^B y\, w\, dx$ and $\int_A^B x\, w' dx$ may be represented as static moments: the former being obtained by loading the arch-points with forces $w \cdot dx$ parallel to the x-axis and multiplying the resulting total moment about this axis by sec τ; the latter being obtained by taking the sum of the moments of the vertical loads $w' \cdot dx$ about the axis of ordinates. All these moments may be determined graphically by means of funicular polygons.

It is first necessary to locate the axes of coordinates. They are determined by Eqs. (32), which may now be written in the form

$$\left.\begin{aligned} t_2 = \frac{\int x' w'' dx}{\int w'' dx}, \quad t_0 = \frac{\int y' w'' dx}{\int w'' dx} \\ \int y' w' dx - \frac{t_1 - t_2}{l} \int x\, w' dx = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (35)$$

The definite integrals occurring here are also to be treated as static moments of the weights w'' and w', and to be obtained graphically by means of funicular polygons. If the arch has either a square or skew symmetry, as defined above, it is only necessary to find t_0.

In carrying out this method of design, the continuous loads w, w', w'' must be replaced by concentrations and the integrals,

consequently, by summations. We therefore divide the arch, commencing at the axis of symmetry, into portions of uniform horizontal length Δx. The coordinates and other quantities pertaining to the different points of division will be marked with the subscripts 0, 1, 2, . . . m, . . . n. In order to evaluate the integrals as accurately as possible by means of finite summations, the loads at the different arch-points will be obtained by applying Simpson's Rule to each three consecutive points; thus:

$$\left.\begin{aligned} W_m &= \tfrac{1}{6}(w_{m-1}+4w_m+w_{m+1}) \\ W'_m &= \tfrac{1}{6}(w'_{m-1}+4w'_m+w'_{m+1}) \\ W''_m &= \tfrac{1}{6}(w''_{m-1}+4w''_m+w''_{m+1}) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (36)$$

If Δx is not an aliquot part of the half-span, the distance of the last ordinate from the end of the span being $\Delta x'$ instead of Δx, then for the last two points we must use

$$\left.\begin{aligned} W_{n-1} &= \tfrac{1}{6}(w_{n-2}+2w_{n-1})+\frac{1}{6}\frac{\Delta x'}{\Delta x}(2w_{n-1}+w_n) \\ W_n &= \tfrac{1}{6}(w_{n-1}+2w_n)\cdot\frac{\Delta x'}{\Delta x} \end{aligned}\right\}, \quad . \quad (36a)$$

and similar expressions for W' and W''.

In the design of any given arch, we first calculate the values of W'' and W' by Eqs. (33) and (36) and use those values in finding the position of the origin O and the inclination of the axis of abscissæ either graphically or analytically by Eqs. (35). In the illustration, Fig. 17, a symmetrical arch is assumed and therefore it is necessary only to determine t_0; this is accomplished by means of the force polygon (a) and the funicular polygon (b) of the weights W''. The ordinates y may then be scaled off or figured and used in calculating the values of W. Since

$\int \frac{y}{I} ds = 0$, we ought to find $\Sigma W = 0$; but, on account of the finite number of the terms in the summation, a small error may

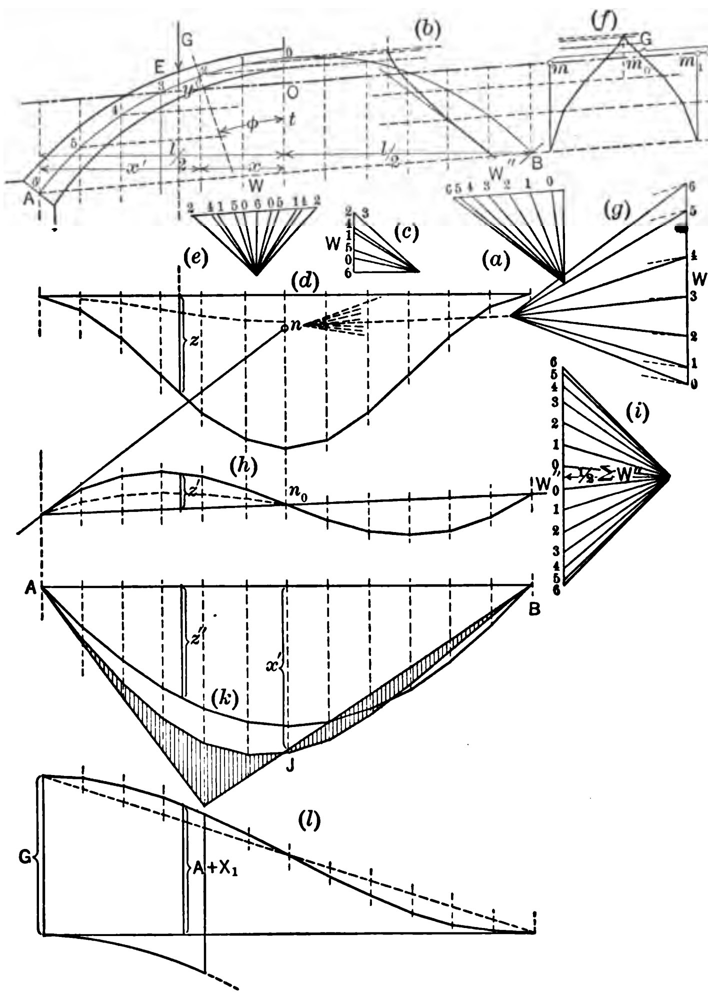

FIG. 17.—Graphic Method for the Hingeless Arch.

be found, which should be distributed proportionately among the different values of W.*

* Instead of using the above described method, viz., dividing the arch into uniform horizontal lengths Δx and figuring the weights W'', W' and W from the

The arch-points must next be loaded with the weights W, first vertically and then parallel to the axis of abscissæ, and the resulting funicular polygons, (d) and (f) (Fig. 17), constructed with the aid of the force polygons (c) and (e) respectively,

reciprocals of the moments of inertia, we may use another method suggested by Professor Dr. R. Schönhöfer ("Statische Untersuchung von Bogen- und Wölbetragwerken unter Anwendung des Verfahrens mit konstanten Bogengrössen," Berlin, 1908, 1911, Wilhelm Ernst & Son). This method consists in dividing the arch into elements Δs proportional to their respective average values of I. For this purpose we draw the rectified arch axis A_1B_1 (Fig. 18), and plot

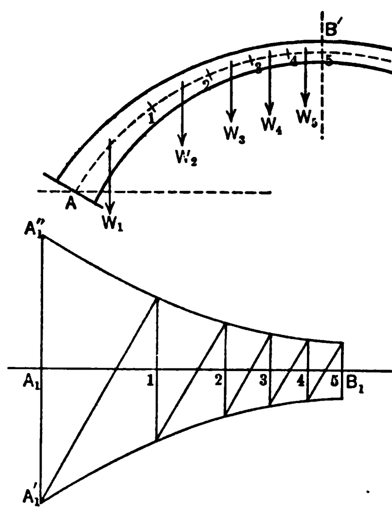

FIG. 18.—Graphic Method for Subdividing the Arch.

thereon the moments of inertia of the different sections as ordinates, half above and half below the axis. Between the two smooth curves drawn through these points, construct a chain of vertical and parallel oblique lines; the resulting points of intersection 1, 2, 3 . . . represent the desired sections of the arch. If the end of the zig-zag line does not strike the end of the I-curve on the first trial, the assumed direction of the oblique lines must be changed or else the error distributed proportionately among all the divisions. After the arch has been divided up in this manner, it is sufficiently accurate to simply take the weights W, W' and W'' at the mid-points of the respective divisions. If the ordinates of these points referred to the chord of the arch are y' and the arch is symmetrical about the center line, the position of the axis of abscissæ is given by $t=\frac{1}{n}\Sigma y'$, where n is the number of divisions in the arch. The weights are then $W=y=y'-t$, $W'=x$, and W'' is constant.

using any pole-distance p_0. (In the force polygon (e), p_0 must be measured vertically.) In the first funicular polygon (d), let z be the ordinate under the load G; and, in the second funicular polygon, let the intercept between the first and last sides of the funicular polygon measured along any line parallel to the axis of abscissæ be $\overline{mm_1} = 2\overline{m_0m}$. Then the horizontal thrust for any load G applied at E is

$$H = \frac{z}{\overline{mm_1} + c} \cdot G,$$

where c denotes the small correction term $c = \frac{v}{r} \cdot \frac{l}{A_0} \cdot \frac{1}{p_0 \cdot \Delta x}$.*
Let us add to $\overline{mm_1}$ the small distance $\overline{m_1m_2}$ representing c to scale and adopt the resulting length $\overline{mm_2}$ as the measure of the load G. Then the ordinate z gives the value of the horizontal thrust H, and the funicular polygon (d) or its circumscribed curve represents the influence line for H.

On account of the assumed symmetry of the arch in this example, points similarly located on the two halves of the arch will have the same values of W and W'', and opposite values of W'; it is therefore sufficient to draw merely one-half of each funicular polygon.

If we conceive the arch-points loaded with the weights W' and, using any pole-distance, construct the corresponding force polygon (g) and the resulting funicular polygon (h), the ordinates z' of the latter polygon measured from its closing chord represent the moments in a simple beam loaded with the same weights, and therefore represent the values of the numerator of the expression for X_1 in Eq. (34). The weights W' on one side of the arch are equal and opposite to those on the other side; the static moment of these weights about the axis of ordinates is therefore given by twice the intercept between

* If the modified method (footnote, p. 45) is used, the correction term will be

$$c = k \cdot \frac{v}{r} \cdot \frac{l}{A_0 \cdot p_0},$$

where k denotes the constant ratio $I : \Delta s$. (—Melan.)

the middle and last sides of the funicular polygon $= 2\overline{n_0 n}$. Consequently

$$X_1 = \frac{z'}{2\overline{n_0 n}} \cdot G.$$

The circumscribed curve of the funicular polygon (*h*) thus represents the influence line for the function X_1, if twice the distance $\overline{nn_0}$ is taken as the unit load G.

Finally, for the loads W'' applied vertically, we draw the force polygon (*i*) and the ordinates of the resulting funicular polygon (*k*) are denoted by z''. If we here choose the pole-distance $= \frac{1}{2}\Sigma W''$, then $X_2 = \frac{1}{2}z'' \cdot G$; in this case the circumscribed curve of the polygon (*k*), measured to twice the scale of lengths, represents the influence line of the function X_2.

In an unsymmetrical arch, the X-axis being inclined to the arch-chord, this method of design becomes somewhat longer but not any more difficult.

With the aid of the above three curves, the *influence lines for moments* and *axial thrusts* may be constructed for each section of the arch. By Eq. (29), we have

$$M = M_b - H \cdot y - X_1 \cdot x - X_2.$$

Hence, retaining the scale of the X_2 curve, i.e., twice the scale of lengths, we add to the ordinates z'' the ordinates z and z' of the H and X_1 curves reduced respectively in the ratios $\frac{2y}{\overline{mm_2}}$ and $\frac{x}{\overline{nn_0}}$. These reduced ordinates are best obtained by redrawing the respective funicular polygons after increasing the pole-distances in the ratios $\frac{\overline{mm_2}}{2y}$ and $\frac{\overline{nn_0}}{x}$. The three sets of ordinates, with due regard to their algebraic signs, are then added with the dividers; the resulting curve is shown in Fig. 17*k* for the point 2. It remains merely to subtract the resultant ordinates from those of the influence line for the moments M_b. As is well known, the latter line is represented

by a triangle; and, retaining the same scale (=twice the scale of lengths), the intercept of the triangle on the mid-vertical should be $=x'=\frac{l}{2}-x$. The shaded ordinates (Fig. 17*k*) thus give the influence values for the moment M at the point 2.

The axial thrust at the section 2, for a load located to the right or to the left of the section, is $N=H\cos\phi+V_1\sin\phi$ or $N=H\cos\phi-V_2\sin\phi$, respectively. If we denote the vertical end reactions of a simple beam by A and B, and those of the hingeless arch by V_1 and V_2, then

$$V_1=A+X_1+H\cdot\tan\tau$$

and

$$V_2=B-X_1-H\cdot\tan\tau,$$

so that

$$\begin{cases} N=H\dfrac{\cos(\phi-\tau)}{\cos\tau}+(A+X_1)\sin\phi, \\ \text{or} \\ N=H\dfrac{\cos(\phi-\tau)}{\cos\tau}-(B-X_1)\sin\phi. \end{cases}$$

The influence lines for $(A+X_1)$ and $(B-X_1)$ are shown in Fig. 17 (*l*). From these, together with the H-curve, the axial thrusts for the given cases of loading may be calculated by multiplying the influence ordinates of these lines by the trigonometric functions expressed in the above equations.

From the above quantities M and N, we can calculate the normal stresses in the section by Eq. (1). To find the *maximum fiber stresses*, we must of course apply the conditions of loading which produce the maximum moments about the core-points. For this purpose we have to construct the moment influence lines for both core-points of each cross-section, substituting the coordinates of these points for x and y in Eq. (29). The determination of the axial forces N may then be omitted. It will be advantageous, however, to adopt the same permissible approximation as was suggested for three-hinged arches,

of assuming a single position of loading for the two extreme fibers of a section; namely that loading which makes the moment about the arch-axis a maximum or minimum. (For instance, for the point 2, the load would have to cover the distance *AJ* or *JB* in Fig. 17.) If the loading is assumed in this manner, the corresponding axial thrust *N* must also be determined.

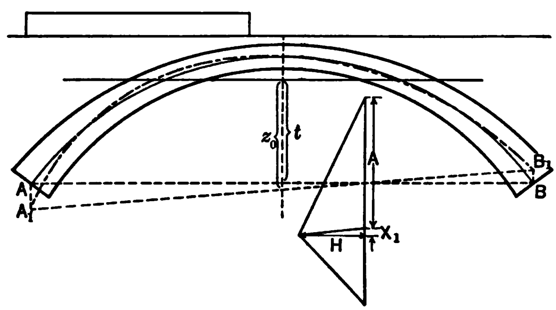

FIG. 19.—Construction of the Line of Resistance.

Knowing the values of H, X_1 and X_2 for any given load position, it is a simple matter to construct the line of resistance. For $z_0 = \frac{X_2}{H}$ determines the position and $\frac{z_1 - z_2}{l} = \frac{X_1}{H}$ the inclination of the closing side A_1B_1 with respect to the axis of abscissæ AB; and the line of resistance is then constructed as the funicular polygon of the loading with H as the pole-distance. (Fig. 19.)

CHAPTER V

EFFECTS OF TEMPERATURE AND DISPLACEMENT OF THE ABUTMENTS

IN a three-hinged arch, a change of temperature produces no stresses, since the elongation of the arch-axis is taken up by the unresisted rise or fall of the crown-hinge.* The same is true of a possible displacement of the abutments. In a hingeless arch, however, the effect of temperature and of a possible yielding of the abutments should be considered, since considerable stresses may thus be produced.

In a hingeless arch or any arch having less than three hinges, a rise of temperature results in an increase, and a fall of temperature in a decrease, of the horizontal thrust H. The appropriate formulæ for design may be deduced from the expanded expression for the work of deformation; this expression for arches of large radius of curvature and with the shearing forces neglected, may be written:

$$W=\frac{1}{2}\int\frac{M^2}{EI}\cdot ds+\int\left(\frac{1}{2}\frac{N}{EA}-\omega T\right)N\cdot ds.$$

Here T denotes the change of temperature and ω the coefficient of expansion.

Adopting the same analytical procedure as in the preceding section, and denoting the unknowns at the vertical

* To be exact, there will occur a somewhat altered distribution of stresses in the arch-sections, for, on account of the altered rise, the line of resistance will have a changed position relative to the arch-axis. But, if the crown deflections are not too great, the above variations of stress will generally be very small and need not be considered, just as we neglect the deformation due to loading in calculating the stresses.

crown-joint by $M_{0,t}$, H_t and S_t, we have the following equations of condition corresponding to Eqs. (15):

$$\left\{\begin{aligned}&\Sigma\frac{M}{I}\cdot\frac{\partial M}{\partial H_t}\cdot\Delta s+\Sigma\frac{N}{A}\cdot\frac{\partial N}{\partial H_t}\cdot\Delta s-E\omega T\cdot\Sigma\frac{\partial N}{\partial H_t}\cdot\Delta s=0.\\&\Sigma\frac{M}{I}\cdot\frac{\partial M}{\partial S_t}\cdot\Delta s+\Sigma\frac{N}{A}\cdot\frac{\partial N}{\partial S_t}\cdot\Delta s-E\omega T\cdot\Sigma\frac{\partial N}{\partial S_t}\cdot\Delta s=0.\\&\Sigma\frac{M}{I}\cdot\frac{\partial M}{\partial M_{0,t}}\cdot\Delta s=0.\end{aligned}\right.$$

Let the arch be weightless and without external loading; then, at the mth joint in the left and right portions of the arch,

$$\left\{\begin{aligned}M_m&=M_{0,t}-H_t\cdot y-S_t\cdot x,\\N_m&=H_t\cos\phi-S_t\sin\phi,\end{aligned}\right.\qquad\left\{\begin{aligned}M'_m&=M_{0,t}-H_t\cdot y'+S_t\cdot x'.\\N'_m&=H_t\cos\phi'+S_t\sin\phi'.\end{aligned}\right.$$

Substituting these values in the preceding equations, denoting by α, β, γ, . . . the coefficients of Eqs. (17) or (16), and by h' the elevation of the right end of the arch above the left, we obtain the following:

$$\left.\begin{aligned}\alpha M_{0,t}+\beta\cdot H_t+\gamma\cdot S_t-E\omega Tl&=0\\\delta M_{0,t}+\gamma\cdot H_t+\epsilon\cdot S_t+E\omega Th'&=0\\\phi M_{0,t}+\alpha\cdot H_t+\delta\cdot S_t&=0\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad (37)$$

If the arch is symmetrical about the crown-section, then γ and $\delta=0$, also $h'=0$. Hence, substituting the coefficients α, β, and ϕ as defined by Eqs. (19), remembering that these have been divided by $\frac{2}{3}\Delta s$, we obtain:

$$H_t=\frac{3}{2}\frac{E\omega Tl}{\Delta s}\cdot\frac{\phi}{\beta\phi-\alpha^2}.\quad . \quad . \quad . \quad . \quad . \quad (38)$$

$$M_{0,t}=-\frac{\alpha}{\phi}\cdot H_t.\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

The reactions due to temperature effect therefore consist of two horizontal forces H_t acting at a distance

$$\frac{\alpha}{\phi}=\frac{\Sigma\frac{y}{I}\cdot\Delta s}{\Sigma\frac{1}{I}\cdot\Delta s}$$

below the crown.

The same result is yielded by the analysis underlying the graphic method, as follows: Under the action of temperature variation, there must be added to Eqs. (22) and (23) the terms

$$\omega T\int_A^B \sin\phi\cdot ds=\omega\cdot T\cdot h',$$

and

$$\omega T\int_A^B \cos\phi\cdot ds=\omega\cdot T\cdot l,$$

respectively. If no other loading is considered and if the coordinate axes defined by Eqs. (32) or (35) are again assumed, we obtain the following values corresponding to Eq. (34):

$$\left.\begin{aligned} H_t&=\frac{E\omega T\cdot l}{\int y\,w\,dx+\frac{v}{r}\frac{l}{A_0}}\\ X_{1,t}&=\frac{E\omega T\cdot h'}{\int x\,w'dx}\\ X_{2,t}&=0\end{aligned}\right\}\quad\ldots\ldots\quad(40)$$

A change of temperature, therefore, gives rise to a pair of end-reactions having a horizontal component H_t and a vertical component $X_{1,t}+H_t\cdot\tan\tau$, and whose line of action passes through the origin of coordinates, O (Fig. 20). The force X_1 becomes zero when the ends AB lie at equal elevations ($h'=0$) and the end-reaction then coincides with the X-axis (Fig. 21).

Introducing the distances $\overline{mm_2}$ and $\overline{n_0n}$ from the graphic construction of Fig. 17, measured to the scale of lengths, we also have

$$\begin{cases} H_t = \dfrac{E \cdot \omega \cdot T \cdot l}{\overline{mm_2} \cdot p_0 \cdot \Delta x}, \\ X_{1,t} = \dfrac{E \cdot \omega \cdot T \cdot h'}{2\overline{nn_0} \cdot p_0' \cdot \Delta x}, \end{cases}$$

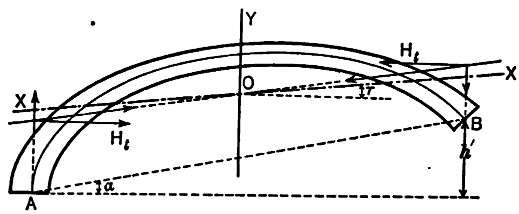

FIG. 20.—Temperature Reactions.

where p_0 and p_0' are the pole-distances of the force polygons (*e*) and (*g*) measured to the same scale as the corresponding weights W and W'.

An exactly similar effect is producible by a displacement of the abutments; for, when the tendency to increase of span

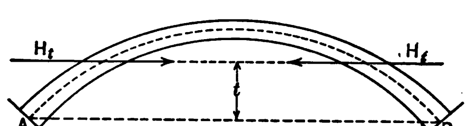

FIG. 21.—Temperature Reactions in a Symmetrical Arch.

$(=\omega Tl)$ due to a rise of temperature is restricted, the same stresses must result as when the abutments are pushed toward each other through $\Delta l = \omega Tl$. Hence, in the above formulæ for H_t, the numerator $E\omega Tl$ must be replaced by $-E \cdot \Delta l$ in order to obtain the effect of an outward displacement of the abutments upon the value of the horizontal thrust.

The resultant forces at any section, due to H_t, are

$$M_t = -H_t \cdot y - X_{1,t} \cdot x,$$

and

$$N_t = H_t \frac{\cos(\phi - \tau)}{\cos \tau} + X_{1,t} \cdot \sin \phi;$$

for the segmental arch with horizontal closing chord,

$$M_t = -H_t \cdot y,$$

and

$$N_t = H_t \cdot \cos \phi.$$

The maximum stresses due to temperature effect usually occur at the crown and at the ends of the arch; a rise in temperature produces tension in the extrados and compression in the intrados at the crown and the reverse at the abutments. A fall in temperature has the opposite effect.

The coefficient of expansion ω has been found for stone and concrete to range from .0000044 to .0000078 per 1° F.; hence, at the mean value, about the same as for steel. As to the fluctuations of temperature, these are not as large in masonry as in metallic structures. Sufficient observations have not been made on masonry constructions. According to the new Austrian specifications for concrete and reinforced concrete highway bridges, a variation of temperature of ±27° F. from the unstressed condition must be provided for, using a coefficient of linear expansion for concrete of .0000066. In the case of bridges having a minimum thickness of concrete exceeding about 30 inches or covered with earth or ballast to an average depth of over 30 inches, the above range of temperature may be reduced to ±18° F. The same range of variation (±27° F.) is prescribed in the Swiss specifications for the calculation of temperature stresses.

CHAPTER VI

NON-VERTICAL LOADS

UNDER certain conditions, the longitudinal forces acting on bridges may require consideration. This occurs in railway bridges built on heavy grades or on rack railways (also on any spans on which the tractive or braking stresses are important). The longitudinal force in such cases may be assumed applied at the crown in a direction parallel to the roadway. We also have to consider the more general problem (as it occurs in roof-arches) of forces acting in oblique directions at the different points of the arch. We resolve such forces into components parallel to the coordinate axes defined by Eqs. (32); so that, in addition to the action of vertical loading, we now have to investigate the effect of loads parallel to the X-axis. For this purpose, as in the preceding investigations, either an analytic or graphic method may be applied. For the present problem, only the analytic method will be taken up in detail.

We first determine the position of the Y-axis (from the first of Eqs. (32), or simply as the axis of symmetry in the case of a symmetrical arch) and the direction of the X-axis. Starting at the section 0 lying in the Y-axis, we divide the arch by sections 1, 2 . . . n, into portions of equal lengths Δs (Fig. 22). The points of the arch are referred by horizontal and vertical coordinates x and y to the crown-point 0 as origin and to an axis of abscissæ drawn through 0 parallel to the X-axis. Now let a force W parallel to the X-axis be applied at the rth section in the left half of the arch. Choosing as unknowns the moment M_0 and the axial components $H' = H \cdot \sec \tau$

and S of the resultant force at the section o, we have, for the left half of the arch:

$$M_0 = M_0;$$

$$M_1 = M_0 - H\cdot y - S\cdot x_1;$$

. . . .

$$M_r = M_0 - H\cdot y_r - S\cdot x_r;$$

. . . .

$$M_m = M_0 - H\cdot y_m + W\cdot\cos\tau(y_m - y_r) - S\cdot x_m;$$

. . . .

$$N_0 = H\cdot\cos\phi_0 - S\cdot\sin\phi_0;$$

. . . .

$$N_r = H\cdot\cos\phi_r - S\cdot\sin\phi_r;$$

. . . .

$$N_m = (H - W\cdot\cos\tau)\cdot\cos\phi_m - (S + W\cdot\sin\tau)\cdot\sin\phi_m;$$

. . . .

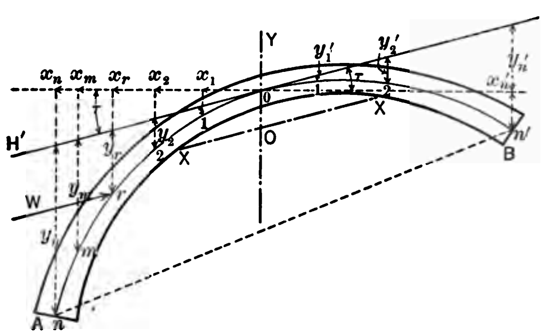

FIG. 22.—The Action of Non-vertical Loads.

for the right half of the arch:

$$M'_1 = M_0 - H\cdot y'_1 + S\cdot x'_1;$$

. . . .

$$M_m = M_0 - H\cdot y'_m + S\cdot x'_m;$$

. . . .

$$N'_m = H\cdot\cos\phi'_m + S\cdot\sin\phi'_m.$$

. . . .

Substituting these expressions in the set of Eqs. (15), taking into account the relations governing the adopted system of coordinates which may be written

$$\sum_0^n \frac{x}{I}\cdot\Delta s-\sum_0^{n'}\frac{x}{I}\cdot\Delta s=0,$$

$$\sum_0^n \frac{xy}{I}\Delta s-\sum^{n'}\frac{xy}{I}\cdot\Delta s-\left(\sum_0^n\frac{\sin\phi\cdot\cos\phi}{A}\cdot\Delta s-\sum_0^{n'}\frac{\sin\phi\cdot\cos\phi}{A}\cdot\Delta s\right)=0,$$

and neglecting the very small terms containing $W\cdot\sin\tau$, we obtain the three equations:

$$\left.\begin{aligned}(\alpha+\alpha')M_0+(\beta+\beta')H-(\beta_r+\alpha_r\cdot y_r)W\cdot\cos\tau&=0\\(\phi+\phi')M_0+(\alpha+\alpha')H-(\phi_r\cdot y_r+\alpha_r)W\cdot\cos\tau&=0\\(\epsilon+\epsilon')S-(\gamma_r+\delta_r\cdot y_r)W\cdot\cos\tau&=0\end{aligned}\right\}\quad.\ .\quad(41)$$

Here the coefficients α, β, ϕ and ϵ are the same as those defined by Eqs. (19) for the left half of the arch, and α', β', ϕ' and ϵ' are the corresponding quantities for the right half of the arch; and the remaining coefficients, with a slight approximation in the case of γ_r, represent the following expressions:

$$\left.\begin{aligned}\alpha_r&=-\left[\frac{y_r}{I_r}+4\frac{y_{r+1}}{I_{r+1}}+2\frac{y_{r+2}}{I_{r+2}}+\ .\ .\ .\ +\frac{y_n}{I_n}\right]\\\beta_r&=\left[\frac{y^2_r}{I_r}+4\frac{y^2_{r+1}}{I_{r+1}}+2\frac{y^2_{r+2}}{I_{r+2}}+\ .\ .\ .\ +\frac{y^2_n}{I_n}\right]\\\phi_r&=\left[\frac{1}{I_r}+4\cdot\frac{1}{I_{r+1}}+2\cdot\frac{1}{I_{r+2}}+\ .\ .\ .\ +\frac{1}{I_n}\right]\\\gamma_r&=\left[\frac{x_ry_r}{I_r}+4\cdot\frac{x_{r+1}\cdot y_{r+1}}{I_{r+1}}+2\frac{x_{r+2}\cdot y_{r+2}}{I_{r+2}}+\ .\ .\ .\ +\frac{x_n\cdot y_n}{I_n}\right]\\\delta_r&=-\left[\frac{x_r}{I_r}+4\frac{x_{r+1}}{I_{r+1}}+2\frac{x_{r+2}}{I_{r+2}}+\ .\ .\ .\ +\frac{x_n}{I_n}\right]\end{aligned}\right\}\ .\quad(42)$$

If the load W is applied at the crown, then

$$\alpha_r=\alpha, \quad \beta_r=\beta, \quad y_r=0;$$

hence

$$\left.\begin{aligned} H'&=H\sec\tau=\frac{\beta(\phi+\phi')-\alpha(\alpha+\alpha')}{(\beta+\beta')(\phi+\phi')-(\alpha+\alpha')^2}W \\ S&=\frac{\gamma}{\epsilon+\epsilon'}W\cdot\cos\tau \\ M_0&=\frac{\alpha\beta'-\alpha'\beta}{(\beta+\beta')(\phi+\phi')-(\alpha+\alpha')^2}W\cdot\cos\tau \end{aligned}\right\} \quad . \quad . \quad (43)$$

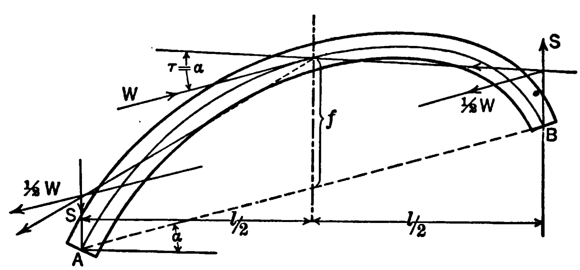

FIG. 23.—Non-vertical Loads on a Symmetrical Arch.

If the arch is symmetrical about the Y-axis (Fig. 23), then

$$\left.\begin{aligned} H'&=H\sec\alpha=\tfrac{1}{2}W \\ S&=\frac{1}{2}\frac{\gamma}{\epsilon}W\cdot\cos\alpha \\ M_0&=0, \quad M_n=-M_n=\tfrac{1}{2}(Wf\cos\alpha-Sl) \\ &=\tfrac{1}{2}W\cdot\cos\alpha\left(f-\frac{1}{2}\frac{\gamma}{\epsilon}l\right) \end{aligned}\right\} \quad . \quad . \quad (44)$$

CHAPTER VII

APPROXIMATE METHODS FOR THE HINGELESS ARCH

We must first note an interesting property of the line of resistance. If, in the third of Eqs. (26), we neglect the very small and practically negligible term containing the axial thrust, there results

$$\int_A^B \frac{M}{I} ds = \int_A^B \frac{M}{I \cdot \cos\phi} \cdot dx = 0.$$

But, as previously shown, $M = H \cdot h$, where h is the vertical intercept between resistance-line and arch-axis; hence the above equation becomes

$$\int_A^B \frac{h}{I \cdot \cos\phi} dx = 0,$$

and, if $I \cdot \cos\phi$ is assumed constant,

$$\int_A^B h \cdot dx = 0.$$

Consequently, if the vertical projection of the moment of inertia of the cross-section is constant, the algebraic sum of the areas included between arch-axis and resistance-line must vanish.

This property, in a certain measure, affords a check on the correctness of the construction of the line of resistance. Even if the moment of inertia $I \cdot \cos\phi$ is not exactly constant but, as is generally the case, increases somewhat toward the abutments, we may still estimate the balancing of the areas after

reducing them approximately in the ratio of $1 : I \cdot \cos \phi$. In this way, for instance, a resistance-line found entirely above or entirely below the axis of the arch would at once be recognized to be incorrect.

In flat segmental arches of moderate span, we may assume the axis to be approximately a parabola and the moment of inertia $I \cdot \cos \phi$ $(=I_0)$ to be constant. With these assumptions, the axis of abscissæ of Fig. 15 is found at a distance

$$t = \frac{1}{l}\int_A^B y' dx = \frac{2}{3}f$$

above the arch-chord. (Here f is the rise of the arch.) We also find

$$\int_A^B y^2 dx = \int_{-\frac{l}{2}}^{+\frac{l}{2}} f^2\left(\frac{1}{3} - 4\frac{x^2}{l^2}\right)^2 dx = \frac{4}{45}f^2 l,$$

and, for a *full-span load* of p per unit length,

$$\int_A^B M_b y dx = \frac{1}{8}p\int_{-\frac{l}{2}}^{+\frac{l}{2}} (l^2 - 4x^2) f\left(\frac{1}{3} - \frac{4x^2}{l^2}\right) dx = \frac{1}{90} p l^3 f.$$

Consequently, by Eq. (31), neglecting the second term of the numerator and using the closely approximate value $\frac{v}{r} = 1$, we have

$$H = \frac{1}{8}\frac{pl^2}{f} \cdot \frac{1}{1 + \frac{45}{4} \cdot \frac{I_0}{A_0 \cdot f^2}} = \frac{1}{8} \cdot \frac{pl^2}{f'}. \quad . \quad . \quad . \quad . \quad (45)$$

Hence, under full uniform loading, the rise of the line of resistance amounts to

$$f' = f\left(1 + \frac{45}{4} \cdot \frac{I_0}{A_0 f^2}\right); \quad . \quad . \quad . \quad . \quad . \quad (46)$$

so that, by the previously established principle concerning the equation of areas between resistance-line and arch-axis, the former curve must lie at a distance $\frac{1}{3}(f'-f)=\frac{15}{4}\frac{I_0}{A_0f}$ above the crown and at $\frac{2}{3}(f'-f)=\frac{15}{2}\cdot\frac{I_0}{A_0f}$ below the ends of the arch-axis. If the depth of rib at the crown is d and the arch consists of homogeneous material, we may substitute for the above intercepts the values $\frac{5}{16}\cdot\frac{d^2}{f}$ and $\frac{5}{8}\cdot\frac{d^2}{f}$.

This position of the line of resistance is, of course, no longer entirely correct if the loading is not perfectly uniform, but increases toward the ends, as is the case with the dead load in arches having spandrel filling.

If the radial depth of the arch is greater at the ends than at the crown, the line of resistance will come somewhat lower in order that the negative areas between the arch and resistance-line may be greater than the positive area above the crown. Nevertheless (for uniform loading or even for dead load if the arch is flat), we may always use the above approximate method for estimating the points through which the line of resistance must pass at the crown and ends of the span, and then treat the rib as a three-hinged arch (Fig. 24) for these cases of loading.

After determining in this way the effect of the dead load H_g, the further investigation for a uniform load p covering *one-half* of the span is accomplished as follows:

The horizontal thrust $H=H_g+\frac{1}{2}H_p=H_g+\frac{1}{16}p\frac{l^2}{f'}$.

Also, by the second of Eqs. (31), if $I\cdot\cos\phi=I_0$ is constant,

$$X_1=\frac{\int_{-\frac{l}{2}}^{+\frac{l}{2}}M_bxdx}{\int_{-\frac{l}{2}}^{+\frac{l}{2}}x^2dx}=\frac{1}{32}pl. \quad . \quad . \quad . \quad . \quad . \quad (47)$$

If the arch had hinged supports, the right end B would receive from the half-load a vertical reaction $\frac{1}{8}pl$. But, on account of the fixedness of the ends, that reaction is reduced by the value of X_1 and hence amounts to $(\frac{1}{8}-\frac{1}{32})pl=\frac{3}{32}pl$. The pole for drawing the half-load resistance-line is therefore to be taken $\frac{3}{32}pl$ above the pole-line OO_1. Since the piercing point of the line of resistance at the crown remains unchanged, the construction of that line is thus determined.

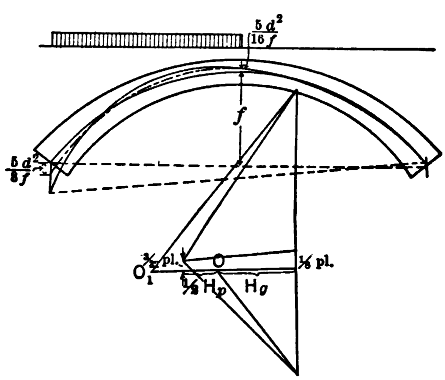

FIG. 24.—Approximate Method for the Hingeless Arch.

Under full loading with p per unit length, the horizontal thrust is $H=H_g+\frac{1}{8}\frac{pl^2}{f'}$, and the moments are given approximately by the following:

$$\left.\begin{aligned} &\text{At the crown, } M_0=\left(H_g+\frac{1}{8}\frac{pl^2}{f'}\right)\cdot\frac{1}{3}(f'-f) \\ &\text{At the ends, } \quad M=-\left(H_g+\frac{1}{8}\frac{pl^2}{f'}\right)\cdot\frac{2}{3}(f'-f) \end{aligned}\right\} \quad . \quad . \quad (48)$$

For the half-load, the approximate expressions are the following:

$$\left.\begin{aligned} &\text{At the end of the loaded side,} \\ &\qquad M=-\frac{1}{64}pl^2-\left(H_g+\frac{1}{16}\frac{pl^2}{f'}\right)\cdot\frac{2}{3}(f'-f). \\ &\text{At the middle of the loaded side,} \\ &\qquad M=\frac{9}{1024}pl^2. \end{aligned}\right\} \quad . \quad . \quad . \quad (49)$$

In arches whose rise-ratio is greater than about $\frac{1}{10}$ to $\frac{1}{8}$, whose form departs markedly from the parabolic, or whose cross-section increases greatly toward the ends of the span, the above approximate rules can no longer be applied.

The horizontal thrust produced by temperature variation has been expressed by Eqs. (40). Substituting for the denominator the approximate value established for flat arches, $\frac{4}{45}\cdot\frac{ff'l}{I_0}$, we obtain,

$$H_t = \frac{45}{4}\frac{E\omega T\cdot I_0}{ff'};$$

and the moment at the crown becomes

$$M_{0,t} = H_t\cdot\frac{1}{3}f = \frac{15}{4}\frac{E\omega T\cdot I_0}{f'}.$$

The resulting extreme fiber stresses at the crown are then approximately

$$f_t = \frac{15}{16}E\omega T\cdot\frac{d^2}{ff'}\left[1\pm2\frac{f}{d}\right],$$

and at the ends,

$$f_t = \frac{15}{16}E\omega T\cdot\frac{d^2}{ff'}\left[1\mp4\frac{f}{d}\right].$$

In masonry and concrete, if we use the mean value $E=2{,}000{,}000$ (lbs. per sq.in.), with $\omega=.0000065$ and $T=\pm27°$ F., then $E\omega T=350$ lbs. per sq.in.

CHAPTER VIII

TWO-HINGED AND ONE-HINGED ARCHES

If hinged joints are provided at the ends of an arch but not at the crown, the system is reduced to single static indeterminateness, namely in the magnitude of the horizontal thrust. To evaluate the latter we make use of the condition that the relative displacement of the two ends of the arch must be zero. From this we derive an equation which is identical with the first of Eqs. (25) if we substitute $\tau=\alpha$ and place the axis of abscissæ in the arch-chord AB. The ordinates y will then represent the heights of the points of the arch-axis above the chord AB.

Noting that $M=M_b-H\cdot y$ in the two-hinged arch, the above equation yields

$$H=\frac{\int_0^l \frac{M_b\cdot y}{I}\cdot ds}{\int^l \frac{y^2}{I}\cdot ds+\frac{v}{r}\int_0^l \frac{dx}{A}}, \quad . \quad . \quad . \quad . \quad . \quad (50)$$

similar to the corresponding equation for the hingeless arch. The influence line for H is then determined by the same graphic method as given for the hingeless arch except that the weights $w=\frac{y}{I}\cdot\frac{ds}{dx}$ must be figured from the ordinates drawn to the arch-chord.

We will not go any further into the study of the two-hinged type, since it is employed principally for steel bridges and is seldom used in masonry or reinforced concrete arches. For

with respect to temperature stresses and the secondary stresses caused by displacement of the abutments or by axial compression, the two-hinged arch does not appear to have any advantage over the hingeless arch, judging by the stresses at the crown, so that the extra cost of providing the hinges yields no corresponding equivalent in the elimination of secondary stresses.

On the other hand, it may sometimes be advisable to provide a hinge at the crown alone, particularly if the limitation of the total height of the structure requires that the depth of rib at the crown should be a minimum while permitting a large increase in the stiffness of the rib at the ends.

The *one-hinged* arch thus obtained is doubly indeterminate, since but one point of the line of resistance is given. Regarding the components H and S of the crown-thrust as the unknowns, the Eqs. (17*a*) and (17*b*) may be used directly for their determination, substituting $M_0=0$. If the arch is *symmetrical* about the crown, we obtain from the first and third of Eqs. (18):

$$\left.\begin{aligned} H&=-\frac{A_1}{\beta} \\ S&=-\frac{A_2}{\epsilon} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Here A_1 and A_2 are defined by Eqs. (20), β and ϵ by Eqs. (19) and the points of the arch are referred to a pair of rectangular axes drawn through the crown-hinge.

The influence lines for the forces H and S may also be found by the graphic method. Let us assume any unsymmetrical form of arch and refer the points of the arch by coordinates x and y measured horizontally and vertically from a pair of axes, X and Y (Fig. 25), the former being provisionally assumed inclined at an unknown angle τ. The resultant pressure at the crown hinge is resolved into two components, the force $H \cdot \sec \tau$ parallel to the X-axis and the vertical shearing force V. If P denotes the external loading between the crown-

hinge and any assumed point E, and if $P \cdot a$ is its moment about E, then for the portion AC,

$$M = P \cdot a - H \cdot y - V \cdot x,$$

$$N = P \cdot \sin \phi + H \cdot \cos (\phi - \tau) - V \cdot \sin \phi.$$

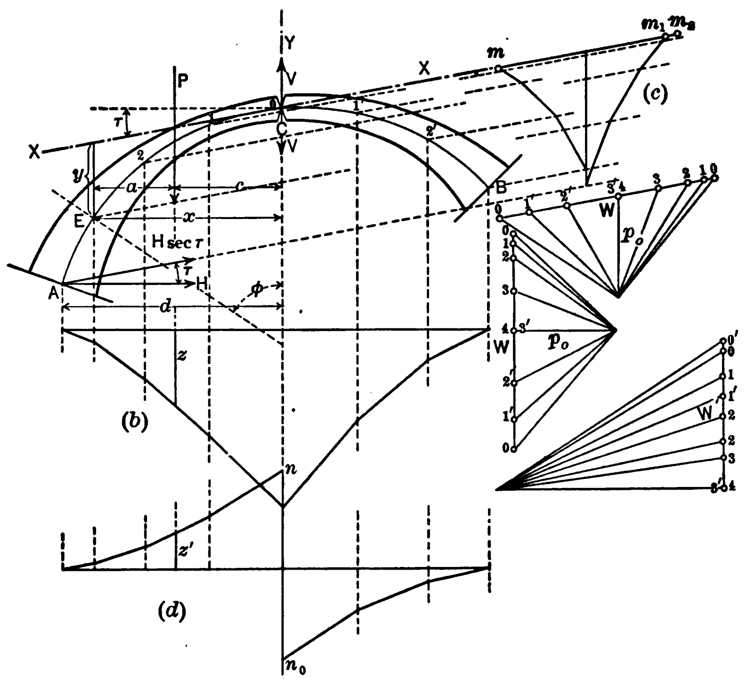

FIG. 25.—Graphic Method for the One-hinged Arch.

The displacements of the point C must have the same values for the two parts of the arch; hence

$$\int_C^A \frac{M}{EI} \cdot \frac{dM}{dH} \cdot ds + \int_C^A \frac{N}{EA} \cdot \frac{dN}{dH} \cdot ds$$
$$= -\int_C^B \frac{M}{EI} \cdot \frac{dM}{dH} \cdot ds - \int_C^B \frac{N}{EA} \cdot \frac{dN}{dH} \cdot ds.$$

$$\int_C^A \frac{M}{EI} \cdot \frac{dM}{dV} \cdot ds + \int^A \frac{N}{EA} \cdot \frac{dN}{dV} \cdot ds$$
$$= -\int_C^B \frac{M}{EI} \cdot \frac{dM}{dV} \cdot ds - \int_C^B \frac{N}{EA} \cdot \frac{dN}{dV} \cdot ds.$$

Substituting the values of M and N, and making the same omissions or approximations in the terms involving the axial thrust as in the preceding investigations, we obtain

$$-\int_C^A \frac{P\cdot a}{I}\cdot yds - \int_C^B \frac{P'\cdot a'}{I}\cdot yds + H\left(\int_C^A \frac{y^2}{I}\cdot ds + \int_C^B \frac{y^2}{I} ds + \int_A^B \frac{dx}{A}\right)$$
$$+V\left(\int_C^A \frac{xy}{I}\cdot ds - \int_C^B \frac{xy}{I}\cdot ds\right) = 0.$$

$$-\int_C^A \frac{P\cdot a}{I}\cdot xds + \int_C^B \frac{P'\cdot a'}{I}\cdot xds + H\left(\int_C^A \frac{xy}{I}\cdot ds - \int_C^B \frac{xy}{I}\cdot ds\right)$$
$$+V\left(\int_C^A \frac{x^2}{I}\cdot ds + \int_C^B \frac{x^2}{I}\cdot ds\right) = 0.$$

We again choose the direction of the X-axis so that

$$\int_C^A \frac{xy}{I}\cdot ds - \int_C^B \frac{xy}{I}\cdot ds = \int_B^A \frac{xy}{I}\cdot ds = 0. \quad . \quad . \quad . \quad (52)$$

This leads to a determination of the axis similar to that given above (page 41) for the hingeless arch.

If the ordinates of the arch-points referred to a horizontal axis drawn through the crown-hinge are denoted by y', then

$$\tan \tau = \frac{\int_B^A \frac{xy'}{I}\cdot ds}{\int_C^A \frac{x^2}{I}\cdot ds + \int_C^B \frac{x^2}{I}\cdot ds}.$$

If the hinge lies in the vertical passing through the center of gravity of the elastic weights $\frac{1}{I}\cdot ds$, so that $\int \frac{x}{I} ds = 0$, then the direction of the X-axis coincides with that of the hingeless arch.

With the X-axis determined as above, we obtain,

$$\left.\begin{aligned} H&=\frac{\int_C^A \frac{P\cdot a}{I}\cdot yds+\int_C^B \frac{P'\cdot a'}{I}\cdot xds}{\int \frac{y^2}{I}\cdot ds+\int \frac{dx}{A}} \\ V&=\frac{\int_C^A \frac{P\cdot a}{I}\cdot xds+\int_C^B \frac{P'\cdot a'}{I}\cdot xds}{\int \frac{x^2}{I}\cdot ds} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (53)$$

In these expressions, the integrals in the denominators are supposed to extend over the entire length of the arch.

If the external loading consists of a single concentration G at a distance c from the crown, the numerators of the above expressions become

$$\int_C^A \frac{P\cdot a}{I}\cdot yds=G\int_C^A \frac{(x-c)y}{I\cdot \cos \phi}\cdot dx,$$

$$\int_C^A \frac{P\cdot a}{I}\cdot xds=G\int_C^A \frac{(x-c)x}{I\cdot \cos \phi}\cdot dx,$$

and the definite integrals of Eqs. (53) can again be represented as the static moments of the weights

$$w=\frac{y}{I\cdot \cos \phi},$$

and

$$w'=\frac{x}{I\cdot \cos \phi},$$

applied as loads along the axis of the arch. For the vertical action of the loads w (or of the concentrations W derived from them by Eq. (36)) we construct the funicular polygon Fig. 25(b), and for the horizontal action the funicular polygon Fig. 25(c). The latter, by its intercept on the axis of abscissæ,

augmented by the small length $\frac{1}{p_0 \cdot \Delta x}\int_0^l \frac{dx}{A}$, gives the denominator of the expression for H. If the length $\overline{mm_2}$ is taken as representing the load G, then the ordinate z of the funicular polygon (b) under the point of loading represents the corresponding value of the horizontal thrust H. Similarly, the ordinate z' of the funicular polygon of the loads w' (or W'), measured under the point of loading, represents the value of V if the distance $\overline{nn_0}$ is taken for the load G. The curves (b) and (d) therefore constitute the influence lines for H and V. The vertical reaction at the left end of the span, for a load located on the left half of the arch, $=G+H\cdot\tan\tau-V$; for a load on the right half of the arch, it is $S=V+H\cdot\tan\tau$.

With the above values, the moments and axial thrusts for the various sections of the arch may readily be determined and their influence lines constructed.

The forces at the crown-hinge produced by a change of temperature are

$$\left.\begin{aligned} H_t\cdot\sec\tau &= \frac{E\omega T\cdot l\cdot\sec^2\tau}{\int\frac{y^2}{I}\cdot ds+\int\frac{dx}{A}} \\ V_t &= \frac{E\omega T\cdot h'}{\int\frac{x^2}{I}\cdot ds} \end{aligned}\right\}, \quad \ldots \quad (54)$$

assuming that the right abutment is higher than the left by an amount h'.

CHAPTER IX

ARCHES WITH ELASTIC ABUTMENTS

THE theory of the hingeless arch developed in Chapter IV presupposes that the end-sections are incapable of displacement or rotation. If this assumption is not fulfilled, the results obtained above will be incorrect, since the expressions in the fundamental Eqs. (22), (23), and (24) are no longer zero, but must be equated to the actual values of the displacements.

We first assume:

(1) *The ends of the arch suffer a horizontal displacement, but no rotation.* Let the resulting increase in length of span be Δl. In this case the first of Eqs. (26) is changed to

$$\int_A^B \frac{My}{I} \cdot ds - \frac{v}{r} \int_A^B \frac{N}{A} \cdot ds = E \cdot \Delta l.$$

The other two equations remain unaltered. In consequence of this, in the subsequent analysis, only the value of H is changed while X_1 and X_2 suffer no change from the expressions given by Eqs. (31) or (34). The altered value of H is

$$H = \frac{\int_A^B M_b \cdot w \cdot dx - E \cdot \Delta l}{\int_A^B y \cdot w \cdot dx + \frac{v}{r} \cdot \frac{l}{A_0}}.$$

If the abutments are elastically deformable in such a manner that a force $H = 1$ displaces the left abutment an

amount δ_1, and the right an amount δ_2, then $\Delta l = H(\delta_1+\delta_2)$; hence

$$H=\frac{\int_A^B M_b\cdot w\cdot dx}{\int_A^B y\cdot w\cdot dx+\frac{v}{r}\cdot\frac{l}{A_0}+E(\delta_1+\delta_2)}\quad . \quad . \quad . \quad . \quad (55)$$

The quantities X_1 and X_2, as already remarked, are unaffected by such displacement if it is not accompanied by a rotation of the abutments. Consequently the graphic method illustrated in Fig. 17 remains unaltered except that the length rcpresenting the unit load G for the H-influence line (Fig. 17(f)) has to be augmented by the amount $\frac{1}{p_0\cdot\Delta x}E(\delta_1+\delta_2)$.

We next assume:

(2) *The abutments are capable of elastic yielding, such that the end-sections of the arch suffer rotation as well as horizontal displacement.*

Let δ_a, δ_b = the horizontal displacements of the left and right end-sections produced by a force $H=1$;

ϕ_a, ϕ_b = the rotation of the end-sections produced by a force $H=1$, or their horizontal displacement produced by a moment $M=1$ acting in the outward direction;

ψ_a, ψ_b = the rotation of the end-sections produced by the moment $M=1$.

These deformations are conditioned by the elasticity of the abutments and are considered as having been determined from their design.

We can bring the problem of the arch with yielding abutments back to the case of the arch with rigid anchorage by imagining each end connected to a vertical pier, anchored at the base, having such height and cross-section that the defor-

mations ϕ and ψ would be produced at the ends of the arch (Fig. 26). We readily obtain

$$\phi=\frac{h^2}{2EI},\quad \psi=\frac{h}{EI},$$

hence

$$\left.\begin{aligned} h&=2\frac{\phi}{\psi} \\ EI&=2\frac{\phi}{\psi^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

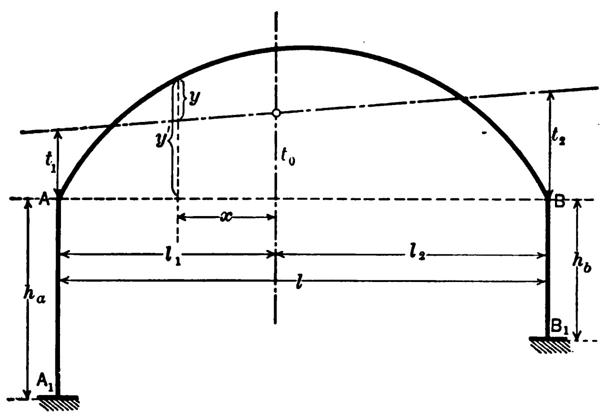

FIG. 26.—Arch with Elastic Abutments.

The upper end of the pier would deflect under a unit horizontal load an amount $\frac{h^3}{3EI}$. But the displacement of the end-section should amount to δ; consequently we must give to each pier an additional horizontal displacement of

$$\delta-\frac{h^3}{3EI}=\delta-\frac{4}{3}\frac{\phi^2}{\psi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

As pointed out above, this displacement affects only the value of the horizontal thrust H.

The arch thus obtained by affixing the elastic piers AA_1 and BB_1 may now be designed directly by the method for the hingeless arch developed in Chapter IV. For the graphic

method we must again first determine the position of the coordinate axes. For this purpose we may apply the defining Eqs. (32) or (35), determining the origin O as the center of gravity of the weights $\frac{ds}{I}$ and the axes as the conjugate diameters of the inertia ellipse of these weights. In this operation, the vertical piers have to be loaded with the weights $\frac{ds}{I_a}$ and $\frac{ds}{I_b}$.

If the arch AB is built symmetrical about the vertical center line, and if the points of the arch are referred to the chord AB by the ordinates y', then the three Eqs. (32) for determining the position of the axes, on substituting $w''=\frac{1}{I\cdot\cos\phi}$ and $w'=\frac{x}{I\cdot\cos\phi}$ become:

$$\left.\begin{aligned} t_0&=\frac{\int_A^B y'w''dx-E(\phi_a+\phi_b)}{\int_A^B w''dx+E(\psi_a+\psi_b)}\\ l_1&=l-l_2=\frac{l}{2}-\frac{E(\psi_a-\psi_b)}{\int_A^B w''dx+E(\psi_a+\psi_b)}\\ &\int_A^B y'\cdot w'\cdot dx-\phi_a\cdot l_1+\phi_b\cdot l_2\\ &-\frac{t_1-t_2}{l}\left[\int_A^B x\cdot w'dx+\psi_a\cdot l_1^2+\psi_b\cdot l_2^2\right]=0 \end{aligned}\right\}\quad . \quad . \quad . \quad (58)$$

If the two abutments, in the case of a symmetrical arch, have equal elastic deformability, then $l_1=l_2=\frac{l}{2}$ and $t_1-t_2=0$, and the X-axis lies at a height

$$t=\frac{\int_A^B y'w''dx-2E\phi}{\int_A^B w''dx+2E\psi}$$

parallel to the chord.

With the axes thus fixed, the quantities H, X_1 and X_2 are calculated from the Eqs. (34) of the arch with fixed ends, but with the integrations now extended over the entire system including the arch and the two attached piers. In the numerators, however, which simply represent the moment curves for a simply supported beam loaded with the weights w, w' and w'', the weights acting on the piers have no influence. These curves are therefore obtained, as before, as the funicular polygons for the vertical loading of the span with the weights w, w', w''.

The denominators of the expressions for H, X_1 and X_2 may be evaluated as follows: Taking into account the displacement $\delta - \frac{h^3}{3EI}$ (Eq. (57)) which must be given to each pier, the horizontal thrust becomes, by Eq. (55),

$$H = \frac{\int_A^B M_b \cdot w\, dx}{\int_{A_1}^{B_1} \frac{y^2}{I} \cdot ds + E(\delta_a + \delta_b) - \frac{h_a^3}{3I_a} - \frac{h_b^3}{3I_b} + \frac{v}{r} \cdot \frac{l}{A_0}}.$$

The integration of the first term carried out for the piers gives

$$\int_{A_1}^{B_1} \frac{y^2}{I} \cdot ds = \int_A^B y \cdot w \cdot dx + \frac{1}{I_a}\left(h_a \cdot t_1^2 + h_a^2 \cdot t_1 + \frac{h_a^3}{3}\right) + \frac{1}{I_b}\left(h_b \cdot t_2^2 + h_b^2 \cdot t_2 + \frac{h_b^3}{3}\right).$$

Substituting this expression, together with $\frac{h^2}{2I} = E \cdot \phi$ and $\frac{h}{I} = E \cdot \psi$, we obtain,

$$H = \frac{\int_A^B M_b \cdot w\, dx}{\int_A^B ywdx + \frac{v}{r} \cdot \frac{l}{A_0} + E(\delta_a + \delta_b + 2\phi_a \cdot t_1 + 2\phi_b \cdot t_2 + \psi_a \cdot t_1^2 + \psi_b \cdot t_2^2)} \tag{59}$$

In the same manner, we may substitute for the denominators of X_1 and X_2:

$$\int_{A_1}^{B_1} \frac{x^2}{I} \cdot ds = \int_A^B x\, w' dx + \frac{a^2 \cdot h_a}{I_a} + \frac{b^2 \cdot h_b}{I_b},$$

$$\int_{A_1}^{B_1} \frac{ds}{I} = \int_A^B w'' dx + \frac{h_a}{I_a} + \frac{h_b}{I_b},$$

so that the expressions become

$$X_1 = \frac{\int_A^B M_b \cdot w' dx}{\int_A^B x\, w' dx + E(\psi_a \cdot a^2 + \psi_b \cdot b^2)}, \quad . \quad . \quad . \quad . \quad (60)$$

$$X_2 = \frac{\int_A^B M_b \cdot w'' dx}{\int_A^B w'' dx + E(\psi_a + \psi_b)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

With the above relations, the influence lines for H, X_1 and X_2 are now simple to construct. We draw the funicular polygons (Fig. 27 (*d*), (*h*), (*k*)) for the elastic weights w, w', w'' applied vertically; and the funicular polygon (Fig. 27 (*f*)) for the loads w applied parallel to the X-axis, to which we add the forces $E(2\phi_a + \psi_a \cdot t_1)$ and $E(2\phi_b + \psi_b \cdot t_2)$ acting at the ends of the arch (force polygon (*e*)). The first and last sides of the funicular polygon (*f*) will then intercept on the X-axis a length

$$\overline{m_0 m_1} = \frac{1}{p_0 \cdot \Delta x}\left[\int_A^B y\, w\, dx + E(2\phi_a \cdot t_1 + \psi_a \cdot t_1^2 + 2\phi_b \cdot t_2 + \psi_b \cdot t_2^2)\right]$$

so that we merely have to add the correction

$$\frac{1}{p_0 \cdot \Delta x}\left[\frac{v}{r} \cdot \frac{l}{A_0} + E(\delta_a + \delta_b)\right] = \overline{m_1 m_2}$$

in order to obtain the full value of the denominator of H. We therefore have, for a concentration $G=1$,

$$H=\frac{z}{\overline{m_0 m_2}}.$$

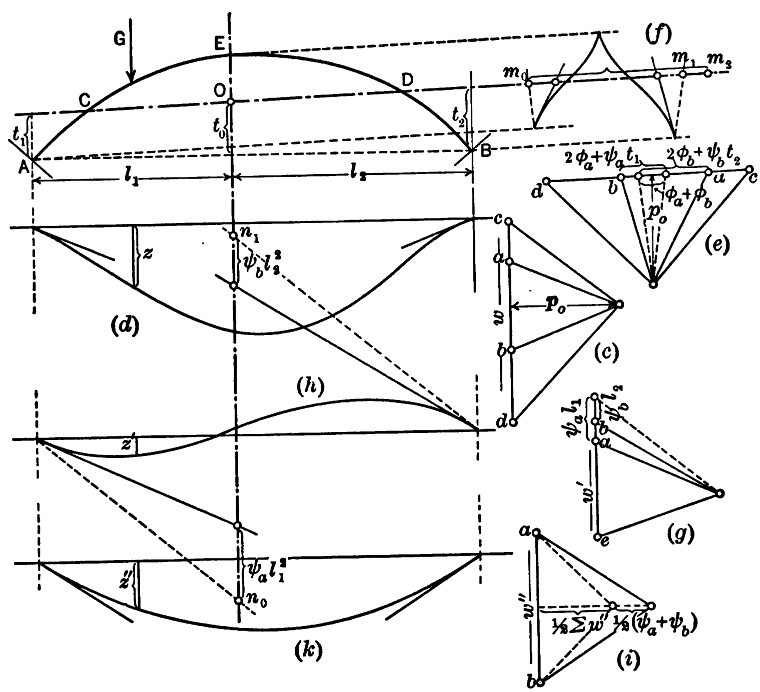

FIG. 27.—Graphic Method for the Arch with Yielding Abutments.

Similarly, to construct the denominator for X_1, the end loads $E\cdot\psi_a\cdot l_1$ and $-E\cdot\psi_b\cdot l_2$ must be added to the elastic weights w' (force polygon (g)). The terminal sides of the funicular polygon (h) will then intercept on the Y-axis the distance

$$\overline{n_0 n_1}=\frac{1}{p_0'\cdot\Delta x}\left[\int_A^B x\,w'dx+E(\psi_a\cdot l_1^2+\psi_b\cdot l_2^2)\right],$$

and we have

$$X_1=\frac{z'}{\overline{n_0 n_1}}.$$

Finally, if we choose as the pole-distance for constructing the funicular polygon (k) one-half the sum of the loads w'' augmented by $\frac{1}{2}E(\psi_a+\psi_b)$, we obtain

$$X_2=\frac{1}{2}z''.$$

This general treatment of the arch with elastic supports may be applied without change to all arches in which the elastic deformability of high abutments or piers has to be considered. We merely have to determine the elastic deformations of the piers, i.e., the quantities δ, ϕ and ψ, producible by a force of unity applied at the springing line.

NOTE. It is apparent that the straight beam or girder framed into elastic supports may be treated as a special case of the above problem, by simply substituting $y'=0$.

CHAPTER X

ARCHES CONTINUOUS OVER SEVERAL SPANS

A SERIES of arches connected with one another and with the intermediate piers constitutes a statically indeterminate system of a degree depending upon the assumed character of the connections as fixed, hinged, or sliding.

It is only in the case of arches connected together with hinges and free to slide on the intermediate piers that the degree of indeterminateness remains the same as for a single arch. If such arches are hinged at the end abutments, we have but the single static indeterminateness of a two-hinged arch. The horizontal thrust, however, is considerably diminished; with n equal arches, it is only $\frac{1}{n}$th of the value of H for a single arch. With this reduction of the arch action, the positive bending moments become considerably larger than in a single arch; and this holds true, although in a somewhat reduced measure, even when the friction at the intermediate piers is taken into account.

If the connections between the arches are rigid (without hinges) and are free to slide on the intermediate piers but hinged at the end abutments, a system of n spans will be n-fold statically indeterminate.*

In actual designs of arches, however, the sliding supports on the intermediate piers are out of the question. Instead we have to assume fixed supports incapable of horizontal displacement except as the result of the elastic deflections

* The theory of this system, which is really a generalized case of the continuous beam, was first presented by *Müller-Breslau.* Wochenblatt für Arch. u. Ing., 1884.

of the piers. If the arches are constructed as independent three-hinged arches (Fig. 28), the pier deflections will cause large crown movements; but, as this slight variation in the rise of the arches may be disregarded, the stresses in the arches will remain unaltered. Loading in any span will have no effect on the stresses in the other spans.

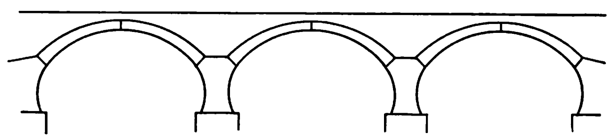

FIG. 28.—Continuous Three-hinged Arches.

Such is not the case if hingeless or two-hinged arches are used; every load then produces stresses in all the arches and piers; but the crown movements and the horizontal thrusts on the intermediate piers are smaller than in the system of three-hinged arches.

The general case of the continuous arch without hinges is represented diagrammatically in Fig. 29. The end-sections

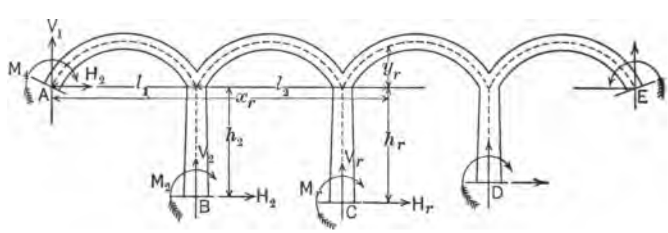

FIG. 29.—Continuous Hingless Arches.

A, *B*, . . . , *E*, of the arches and piers are considered as fixed against sliding or rotation. At each of these points external reactions must be applied, consisting of a horizontal and a vertical force and a bending moment. With n spans we thus have $3(n+1)$ unknowns, three of which are eliminated by applying the conditions of static equilibrium. Consequently there remain $3n$ statically indeterminate quantities.

The solution of this problem, although quite tedious on account of the large number of unknowns, involves no special difficulty since we have only to apply the general method for the design of statically indeterminate systems. For this purpose we write out the expressions for the moment and axial thrust in the sections of each arch and pier; thus, for a section in the rth arch,

$$\left.\begin{aligned} M=&M_1+M_2+\ldots+M_r-H_1y_r-H_2(y_r+h_2)-\ldots \\ &-H_r(y_r+h_r)+V_1\cdot x_r+V_2(x_r-l_1)+\ldots \\ &+V_r(x_r-l_1-l_2-\ldots l_{r-1})-\mathfrak{M}_r \\ N=&(H_1+H_2+\ldots+H_r)\cdot\cos\phi_r+\ldots \\ &+(V_1+V_2+\ldots+V_r)\sin\phi_r-(\overset{r}{\Sigma}P)\cdot\sin\phi_r \end{aligned}\right\} \quad . \quad (62)$$

Here x_r, y_r denote the coordinates of the arch-point referred to A; h_1, h_2, ... = the heights of the piers; $\overset{r}{\Sigma}P$ = the sum of all the loads to the left of the section, $\mathfrak{M}_r$ = their moment about the section. Similarly, for a section taken in any pier at a height x_p,

$$\left.\begin{aligned} M&=M_r-H_r\cdot x_p \\ N&=V_r \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (62a)$$

The equations for determining the $3n$ unknowns are then obtained from the derivatives of the work of deformation $W=\frac{1}{2}\int\frac{M^2}{EI}\cdot ds+\frac{1}{2}\int\left(\frac{N}{EA}+2\omega T\right)N\cdot ds$, namely:

$$\left.\begin{array}{lll} \dfrac{\partial W}{\partial M_1}=0; & \dfrac{\partial W}{\partial M_2}=0 \ \ldots & \dfrac{\partial W}{\partial M_n}=0 \\ \dfrac{\partial W}{\partial H_1}=0: & \dfrac{\partial W}{\partial H_2}=0 \ \ldots & \dfrac{\partial W}{\partial H_n}=0 \\ \dfrac{\partial W}{\partial V_1}=0\cdot & \dfrac{\partial W}{\partial V_2}=0 \ \ldots & \dfrac{\partial W}{\partial V_n}=0 \end{array}\right\} \quad . \quad . \quad (63)$$

After the reactions are calculated from these equations, the moments and axial pressures at the various sections are given by Eqs. (62) and (62*a*).

If only one span is loaded (Fig. 30), the effect of this loading will extend principally over the span itself and the two immediately adjacent. With this simplification there remain only 9 unknowns to be considered. A further reduction is effected if we take advantage of the permissible approximation of considering the vertical reactions V_{r-1} and V_{r+2} in the outside piers and the corresponding end moments M_{r-1} and M_{r+2} as equal to zero. The degree of static indeterminateness is then reduced to 5. Nevertheless this treatment of the continuous arch will always involve tedious computations.

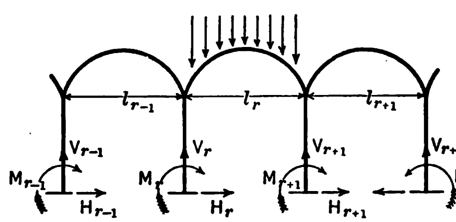

FIG. 30.—Continuous Arches with only One Span Loaded.

A *simpler* solution, *sufficiently accurate* for concrete arches, consists in treating each span separately as an arch with elastic abutments and applying thereto the theory developed in the preceding chapter. There remains only to determine as correctly as possible the elastic deformations δ, ϕ, and ψ at the ends of each arch.

Let AB be the span under consideration (Fig. 31), connected on the left with the arches I, II, . . . on the piers P_1, P_2, P_3, ... Assuming first that the pier P_2 is unyielding and that the arch I is free to slide on pier P_1 but hinged to arch II and pier P_2, a horizontal force of unity applied at A would produce a displacement

$$\delta_{A_1} = \int^{\mathrm{I}} \frac{y^2}{EI} \cdot ds.$$

Similarly a unit horizontal force applied to the top of pier P_1 would produce in the pier a deflection δ_{P_1}. In reality, however, the arch and pier are connected and must deflect together at A; consequently the force unity must be shared by both; it will divide itself between the two in the inverse ratio of the respective deflections δ_{A_1} and δ_{P_1}, so that the force acting on the arch will be only $H_1 = \frac{\delta_{P_1}}{\delta_{A_1}+\delta_{P_1}}$, and the resulting displacement of A will be $\frac{\delta_{A_1}\cdot\delta_{P_1}}{\delta_{A_1}+\delta_{P_1}}$. We next assume

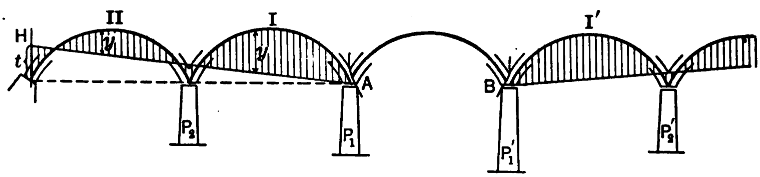

FIG. 31.—Simplified Method for Continuous Arches.

pier P_3 as unyielding, and consider the force H_1 transmitted to arch II and pier P_2; the resulting deflection will be

$$\frac{\delta_{P_2}\cdot\delta_{A_2}}{\delta_{A_2}+\delta_{P_2}}\cdot\frac{\delta_{P_1}}{\delta_{A_1}+\delta_{P_1}},$$

where δ_{A_2} and δ_{P_2} represent the deflections producible in arch II and pier P_2 by a unit force acting on each. Continuing in this manner to the end abutment, we obtain for the total displacement of A,

$$\delta_a = \frac{\delta_{P_1}}{\delta_{A_1}+\delta_{P_1}}\left\{\delta_{A_1}+\frac{\delta_{P_2}}{\delta_{A_2}+\delta_{P_2}}\left[\delta_{A_2}+\frac{\delta_{P_3}}{\delta_{A_3}+\delta_{P_3}}(\delta_{A_3}+\ldots)\right]\right\}. \quad (64)$$

However great the number of spans, it will always be sufficiently accurate to extend the computation of this deflection over but one or, at most, two spans.

The displacement constants for the arches are to be calculated from $\delta_A = \int\frac{y^2}{EI}\cdot ds$. The ordinates y must be measured in

each arch from a line whose height above the springing joints is proportional to the moments of constraint at those points. We may therefore draw it as a straight line AH intercepting on the last (fixed) pier the distance t (see Eq. (35)).

The constant ϕ, denoting the rotation of the section A producible by a unit horizontal force, or the horizontal displacement producible by a unit moment, is calculated from $\phi = -\int \frac{y}{EI} \cdot ds$. On account of the connection between arch and pier, this quantity will of course be modified; but the exact effect is difficult to determine. However, if the arch and pier are not built monolithic, we may safely assume the connection to be hinged, thus reducing the difficulty by precluding a transfer of bending moment (Fig. 32). With this assumption, when a unit force acts to produce a joint deflection of arch and pier, the part of that thrust affecting the arch will be

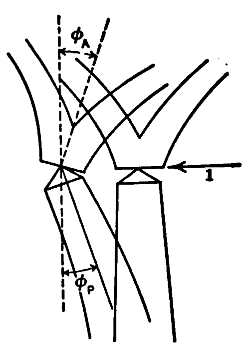

FIG. 32.

$$H_1 = \frac{\delta_{P_1}}{\delta_{A_1} + \delta_{P_1}},$$

and the rotation of the end section will be

$$-\frac{\delta_{P_1}}{\delta_{A_1} + \delta_{P_1}} \int^{\mathrm{I}} \frac{yds}{EI}.$$

The arch thrust H_1 is transmitted to the left end of arch I; then, as before, only the fractional part $H_1 \frac{\delta_{P_2}}{\delta_{A_2} + \delta_{P_2}}$ acts on arch II, so that the rotation of the arch above pier II will be

$$-\frac{\delta_{P_1}}{\delta_{A_1} + \delta_{P_1}} \cdot \frac{\delta_{P_2}}{\delta_{A_2} + \delta_{P_2}} \int^{\mathrm{II}} \frac{yds}{EI}.$$

The total rotation of the section A will then be given by

$$\phi_a = -\frac{\delta_{P_1}}{\delta_{A_1}+\delta_{P_1}}\left[\int^{\mathrm{I}}\frac{yds}{EI}+\frac{\delta_{P_2}}{\delta_{A_2}+\delta_{P_2}}\left(\int^{\mathrm{II}}\frac{yds}{EI}+\ .\ .\ .\right)\right]. \quad (65)$$

The constant ψ, denoting the end rotation producible by a moment $M=1$ acting on the arch, is given by $\psi=\int\frac{ds}{EI}$. If we assume, as above, that there is no transfer of bending moment from arch to pier, we should have

$$\psi_a=\int^{\mathrm{I}}\frac{ds}{EI}+\int^{\mathrm{II}}\frac{ds}{EI}+\ .\ .\ .=\psi_{A_1}+\psi_{A_2}+\ .\ .\ . \quad . \quad (66)$$

If, on the other hand, we count on the piers relieving the bending moment, and if ψ_P denotes the rotation of the top of the pier producible by a moment $M=1$, then the joint rotation of arch and pier $=\frac{\psi_P\cdot\psi_A}{\psi_A+\psi_P}$; and we obtain analogous to the expression for the displacement δ_a,

$$\psi_a=\frac{\psi_{P_1}}{\psi_{A_1}+\psi_{P_1}}\left[\psi_{A_1}+\frac{\psi_{P_2}}{\psi_{A_2}+\psi_{P_2}}(\psi_{A_2}+\ .\ .\ .\ )\right]. \quad . \quad (66a)$$

In like manner we may calculate the quantities δ_b, ϕ_b and ψ_b for the spans on the right. The arch AB may then be designed by the method developed in Chapter IX.

A simpler (but less accurate) method of determining the effect of elastic intermediate piers under a series of arches is as follows:

In the three arches of Fig. 33, unequal in span and loading, assume the horizontal thrust to have been calculated for any given condition of loading on the assumption of rigid, immovable supports. Let this thrust be H_0 in the middle arch and H'_0 and H''_0 in the side spans. The intersection of the arch reactions, if the end-points are at a uniform elevation, will fall very close to the axes of the piers. Let h_1 and

h_2 be the heights of these intersection points above the pier bases which are assumed as fixed. We adopt a constant or mean moment of inertia I_1 and coefficient of elasticity E_1 for the piers, and assume the outside piers as rigid, thus providing immovable abutments. The horizontal thrusts acting on the intermediate piers will cause them to deflect amounts $\Delta s'$ and $\Delta s''$, thereby changing the arch thrusts to the values

$$H'=H'_0+\Delta H', \quad H=H_0+\Delta H, \quad H''=H''_0+\Delta H''. \quad . \quad (67)$$

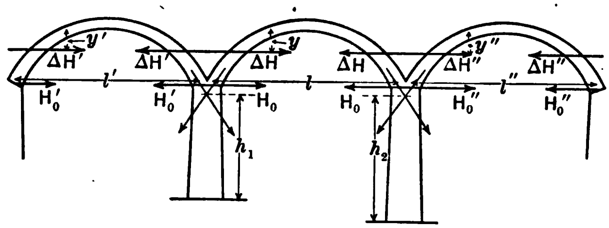

FIG. 33.—Approximate Method for Continuous Arches.

If these deflections are directed outward, so as to reduce the side spans, we have

$$\Delta H'=+\frac{E\cdot\Delta s'}{C'}, \quad \Delta H=-\frac{E(\Delta s'+\Delta s'')}{C}, \quad \Delta H''=+\frac{E\cdot\Delta s''}{C''}. \quad (68)$$

Here C', C, C'' are the familiar arch functions:

$$C'=\int_0^{l'}\frac{y'^2\cdot ds}{I'}+\frac{v'}{r'}\cdot\int_0^{l'}\frac{dx}{A'}, \quad C=\int_0^{l}\frac{y^2\cdot ds}{I}+\frac{v}{r}\int_0^{l}\frac{dx}{A},$$

$$C''=\int_0^{l''}\frac{y''^2\cdot ds}{I''}+\frac{v''}{r''}\int_0^{l''}\frac{dx}{A''}.$$

The deflections of the piers are given by

$$\Delta s'=(H-H')\frac{h_1^3}{3E_1I_1} \quad \text{and} \quad \Delta s''=(H-H'')\frac{h_2^3}{3E_1I_1},$$

or, employing an abbreviation of notation,

$$\Delta s'=k_1(H-H'), \quad \Delta s''=k_2(H-H'').$$

Combining these with Eqs. (67) and (68), we obtain

$$\left.\begin{aligned}
&\Delta s'\left[\left(1+\frac{C+C'}{CC'}E\cdot k_1\right)\left(1+\frac{C+C''}{CC''}E\cdot k_2\right)-\frac{E^2\cdot k_1k_2}{C^2}\right]\\
&\quad=(H_0-H'_0)\left(1+\frac{C+C''}{CC''}Ek_2\right)k_1-(H_0-H''_0)\frac{Ek_1k_2}{C}\\
&\Delta s''\left[\left(1+\frac{C+C'}{CC'}E\cdot k_1\right)\left(1+\frac{C+C''}{CC''}E\cdot k_2\right)-\frac{E^2\cdot k_1k_2}{C^2}\right]\\
&\quad=(H_0-H''_0)\left(1+\frac{C+C'}{CC'}Ek_1\right)k_2-(H_0-H'_0)\frac{Ek_1k_2}{C}
\end{aligned}\right\}. \tag{69}$$

With the values of $\Delta s'$ and $\Delta s''$ determined by Eqs. (69), we obtain the corrections to be applied to the horizontal thrusts, viz., $\Delta H'$, ΔH and $\Delta H''$ from Eqs. (68). These forces ΔH may be assumed to act along the axis located at the height t (or t', t'') above the arch chord, and the resulting extra moments and stresses in the arches can then be calculated. These will have their maximum value for the condition of loading giving the greatest differences between the horizontal thrusts H_0, H'_0 and H''_0.

CHAPTER XI

DETERMINING THE BEST CURVE FOR ANY ARCH

As previously pointed out (Chapter III, page 20) in order that an arch may be built with minimum thickness, the axis of the arch must coincide as closely as practicable with the mean of the lines of resistance for the various conditions of loading. This mean line of resistance is obtained in the case of the three-hinged arch, and approximately in the hingeless arch, by loading the entire span with one-half of the total live load uniformly distributed.

The determination of the best arch curve is thus reduced to finding the above line of resistance. We may assume its span l and rise f as fixed by the requirements of the structure. Thus, if the clear span of the bridge and the elevations of the roadway and abutments are specified, and if the depth of filling above the crown is assumed and the thickness of the arch rib is chosen on the basis of formulæ to be presented later, the piercing points of the line of resistance at the crown and end-sections are fixed approximately and the distances l and f are thereby determined.

1. Exact Method. (Fig. 34.)

Let d_0 = the thickness of the rib at the crown;

u_0 = the depth of filling at the crown;

γ, γ_1 = the respective densities of the arch concrete and the filling (lbs. per cu.ft.);

p = the live load, expressed as equivalent uniform load in lbs. per sq.ft.

The intensity of loading at the crown (in lbs. per sq.ft.) corresponding to the mean line of resistance will then be

$$q_0 = \gamma \cdot d_0 + \gamma_1 \cdot u_0 + \frac{p}{2}. \quad . \quad . \quad . \quad . \quad . \quad (70)$$

We assume that the thickness of rib increases from the crown toward the abutments in proportion to the secant of the included arc, so that all the sections have the same vertical projection; hence

$$d = d_0 \cdot \sec \phi.$$

The roadway or the top of the filling is assumed to rise from the ends to the crown along a parabolic curve of height h, so

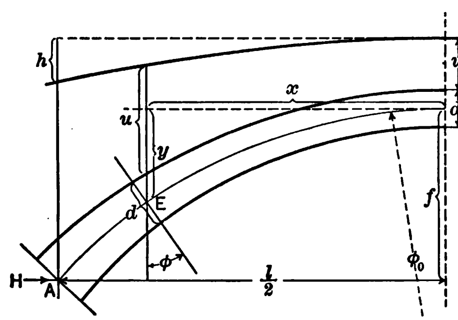

FIG. 34.—Determination of the Arch Curve. Exact Method.

that its ordinates measured from the horizontal crown tangent are given by

$$\frac{4h}{l^2} \cdot x^2 = \beta \cdot x^2.$$

The intensity of loading at any point E will then be

$$\begin{aligned} q &= \gamma \cdot d \cdot \sec \phi + \gamma_1 \cdot u + \frac{p}{2} \\ &= \gamma \cdot d_0 \cdot \sec^2 \phi + \gamma_1 (u_0 + y - \tfrac{1}{2} d_0 \cdot \tan^2 \phi - \beta x^2) + \frac{p}{2} \\ &= \gamma d_0 + \gamma_1 u_0 + \frac{p}{2} + \gamma_1 y + (\gamma - \tfrac{1}{2}\gamma_1) d_0 \tan^2 \phi - \gamma_1 \beta x^2. \end{aligned}$$

As a close approximation, we may consider the arch axis a parabola, so that $\dfrac{dy}{dx} = \tan \phi = \dfrac{8f}{l^2} \cdot x$ and $\tan^2 \phi = \dfrac{64 f^2}{l^4} \cdot x^2$. We then obtain

$$q = q_0 + \gamma_1 y + \left[\left(\gamma - \frac{1}{2} \gamma_1 \right) \frac{64 f^2 \cdot d_0}{l^4} - \gamma_1 \beta \right] \cdot x^2.$$

If H is the horizontal thrust, the differential equation of the line of resistance is *

$$\frac{d^2y}{dx^2}=\frac{q}{H}.$$

Substituting the value of q and introducing the abbreviation

$$\left[\left(\gamma-\frac{1}{2}\gamma_1\right)\frac{64f^2d_0}{l^4}-\gamma_1\beta\right]=w_1,$$

we obtain

$$H\frac{d^2y}{dx^2}=q_0+\gamma_1 y+w_1x^2.$$

The integration of this equation yields

$$y=k_1\cdot e^{+x\sqrt{\frac{\gamma_1}{H}}}+k_2\cdot e^{-x\sqrt{\frac{\gamma_1}{H}}}-\frac{w_1}{\gamma_1}\cdot x^2-\frac{2w_1}{\gamma_1^2}H-\frac{q_0}{\gamma_1}.$$

To find the constants of integration, k_1 and k_2, we apply the conditions that for $x=0$, $y=0$; and, with symmetrical loading, for equal values of $\pm x$, the ordinates y must be equal. We thus find

$$k_1=k_2=\frac{w_1}{\gamma_1^2}H+\frac{q_0}{2\gamma_1}.$$

The equation of the arch curve then becomes

$$y=\frac{1}{\gamma_1}\left(\frac{w_1}{\gamma_1}H+\frac{1}{2}q_0\right)\left(e^{x\sqrt{\frac{\gamma_1}{H}}}+e^{-x\sqrt{\frac{\gamma_1}{H}}}-2\right)-\frac{w_1}{\gamma_1}x^2.$$

Expanding into a series, which is sufficiently convergent to permit stopping at four terms, we finally obtain

$$y=\left[\left(\frac{1}{2}q_0+\frac{w_1}{\gamma_1}\cdot H\right)\left(1+\frac{x^2}{12}\cdot\frac{\gamma_1}{H}\right)\frac{\gamma_1}{H}-w_1\right]\frac{x^2}{\gamma_1}$$

$$=\left[\frac{1}{2}q_0\left(1+\frac{x^2}{12}\cdot\frac{\gamma_1}{H}\right)+w_1\frac{x^2}{12}\right]\frac{x^2}{H}.$$

Applying this equation to the ends of the arch, we obtain

$$f=\frac{q_0l^2}{8H}\left(1+\frac{l^2}{48}\cdot\frac{\gamma_1}{H}\right)+\frac{w_1l^4}{192H}.$$

* See proof on page 96.

Substituting in this equation the abbreviation

$$q''=\frac{w_1 l^2}{24}=\left[\frac{8}{3}\frac{f^2}{l^2}\cdot d_0\left(\gamma-\frac{1}{2}\gamma_1\right)-\frac{1}{6}h\gamma_1\right], \quad . \quad . \quad . \quad (71)$$

and solving for the horizontal thrust, we find

$$H=(q_0+q'')\frac{l^2}{16f}\left[1+\sqrt{1+\frac{2}{3}\frac{q_0}{(q_0+q'')^2}\gamma_1 f}\right]. \quad . \quad (72)$$

Finally, substituting,

$$m=\frac{\gamma_1 q_0}{12H}+\frac{4q''}{l^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

we obtain the *ordinates for the arch curve,*

$$y=(q_0+mx^2)\frac{x^2}{2H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

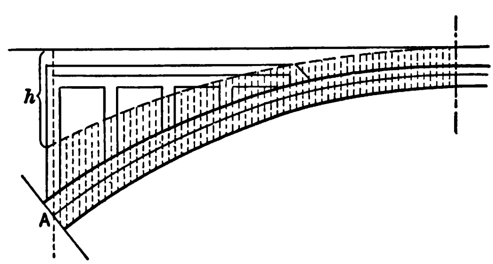

FIG. 35.—Load Profile for Open Spandrels.

These formulæ may be employed to determine the arch curve even if the spandrel filling is not solid but is provided with openings and the roadway is supported on the arch by means of spandrel piers. In such case, with a provisional assumption of the arch curve, the weights on the several piers are figured and the heights of equivalent continuously distributed loading of density γ or γ_1 are plotted (Fig. 35). The resulting boundary of the spandrel area may be again regarded as a parabola with the rise h, and the computations may then be carried out with the above formulæ.

The radius of curvature at the crown of the arch is given by *

$$r_0=\frac{H}{q_0}=\left(1+\frac{q''}{q_0}\right)\frac{l^2}{16f}\left[1+\sqrt{1+\frac{2}{3}\frac{q_0}{(q_0+q'')^2}\gamma_1 f}\right]. \quad . \quad (75)$$

2. *Simplified Method.*—A simpler but less accurate determination of the resistance curve may be developed as follows:

Assume that the loading under consideration (dead weight g of the arch and filling plus one-half the live load p) is so distributed over the span that a graph (Fig. 36) representing the intensity of loading at each point may be replaced by a

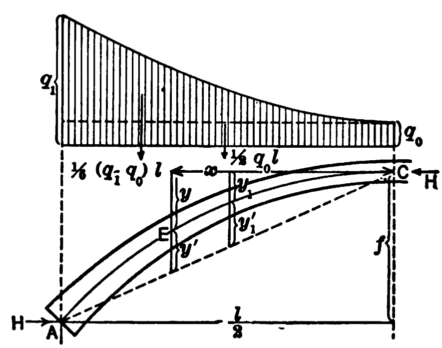

FIG. 36.—Determination of the Arch Curve. Simplified Method.

parabola whose axis coincides with the axis of symmetry of the arch. This parabolic distribution of loading is a very close assumption if the rise-ratio is not too great, since the arch curve is then also practically a parabola.

Let

q_0 = the loading per unit area at the crown $=g_0+\frac{1}{2}p$;

q_1 = the loading per unit area at the springing $=g_1+\frac{1}{2}p$.

Assume that the points at the crown and ends of the arch through which the resistance-line must pass are known; let the corresponding span and rise be denoted by l and f.

* See proof on page 96.

The loading on either half of the span then consists of a part $\frac{1}{2}q_0l$ with its resultant at $\frac{1}{4}l$ from the springing, and a part $\frac{1}{6}(q_1-q_0)l$ with its resultant at $\frac{1}{8}l$ from the springing. The horizontal thrust is then found by equating the moments about point A:

$$Hf=\tfrac{1}{8}q_0l^2+\tfrac{1}{48}(q_1-q_0)l^2;$$

hence,

$$H=\frac{1}{48}(5q_0+q_1)\frac{l^2}{f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

To find the line of resistance at any point E, we note that its intercept y' above the chord AC multiplied by H represents the bending moment in a similarly loaded beam of span $\frac{l}{2}$. The moment of the right end-reaction is

$$[\tfrac{1}{4}q_0l+\tfrac{1}{24}(q_1-q_0)l]x,$$

and the moment of the load between C and E about E is

$$\frac{1}{2}q_0x^2+\frac{1}{3}(q_1-q_0)\frac{x^4}{l^2};$$

hence the resultant moment is

$$Hy'=\frac{1}{24}(5q_0+q_1)lx-\frac{1}{6}\left[3q_0+2(q_1-q_0)\frac{x^2}{l^2}\right]x^2.$$

Substituting the value of H derived above, we obtain

$$y'=2\frac{x}{l}f-8\frac{3q_0+2(q_1+q_0)\frac{x^2}{l^2}}{(5q_0+q_1)}\cdot\frac{x^2}{l^2}\cdot f.$$

If we measure from the horizontal tangent at the crown, the *ordinate* of the point E will then be

$$y=\frac{8f}{(5q_0+q_1)}\left[3q_0+2(q_1-q_0)\frac{x^2}{l^2}\right]\frac{x^2}{l^2}. \quad . \quad . \quad . \quad (77)$$

At the middle of the haunches, i.e., at $x=\frac{1}{4}l$, the ordinate will be

$$y_1=\frac{23q_0+q_1}{16(5q_0+q_1)}\cdot f. \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

The *radius of curvature*, r_0, of the line of resistance at the crown is found from Eq. (81), $H=q_0r_0$:

$$r_0=\frac{1}{6}\left(5+\frac{q_1}{q_0}\right)\frac{l^2}{8f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

This may be written more briefly

$$r_0=\frac{1}{8}\frac{l^2}{f_0},$$

where f_0 is a reduced arch-rise defined by

$$f_0=\frac{6q_0}{5q_0+q_1}\cdot f. \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

It will generally be sufficient to determine merely the radius r_0 at the crown and the ordinate y_1 at the quarter-points, for with these we can draw a five-centered curve as an approximate curve for the arch.

If the loading is uniform over the whole span, so that $q_1=q_0$, there results $y_1=\frac{1}{4}f$, which represents a parabola. With increasing difference between the loads q_0 and q_1 at crown and springing, the haunches become higher (i.e., y' increases) and the axis of the arch first approximates a circular arc and then becomes a compound ("basket-handle") curve with curvature increasing toward the springing.

When the crown and end points of an arch are given, we first calculate the required thickness by the formulæ in the following chapter, thus determining the loads q_0 and q'' or q_1; and then compute the arch ordinates either by the exact Eqs. (71) to (74) or by the approximate Eqs. (77) to (79). With the axis of the arch thus determined, construct or calculate the line of resistance for the revised dead load plus a

uniform load of $\frac{1}{2}p$; if the arch has fixed ends, the approximate formulæ of Chapter VII may be used to give the controlling points of the line of resistance at the crown and springing. If this line of resistance differs considerably from the assumed arch curve, that curve is corrected to fit the resistance line. We may then proceed with the exact static investigation of the arch.

EXAMPLE. It is required to find the resistance curve for an arch with the following data: $l=60$ ft., $f=15$ ft., $d_0=1.5$ ft., $u_0=1.5$ ft., $p=200$ lbs. per sq.ft., $\gamma=150$ lbs. per cu.ft, $\gamma_1=120$ lbs. per cu.ft. The roadway is horizontal and the spandrel filling is solid.

1. Applying the Eqs. (71)–(74) of the *exact method*, we obtain:

$$q_0=1.5\times150+1.5\times120+100=505 \text{ lbs. per sq.ft.};$$

$$q''=\frac{8}{3}\frac{\overline{15}^2}{\overline{60}^2}\,1.5(150-60)=22.5 \text{ lbs. per sq.ft.};$$

$$H=527.5\frac{\overline{60}^2}{240}\left[1+\sqrt{1+\frac{2}{3}\frac{505}{\overline{527.5}^2}\cdot120\cdot15}\right]=22050 \text{ lbs.}$$

$$r=\frac{120\times505}{12\times22050}+\frac{90}{3600}=0.254;$$

$$y=(505+0.254x^2)\frac{x^2}{44100}.$$

For $x=\frac{l}{4}=15$ ft., $y_1=2.87$ ft.

The radius of curvature at the crown will be

$$r_0=\frac{H}{q_0}=43.6 \text{ ft.}$$

2. Applying the Eqs. (76)–(78) of the *approximate method*, we obtain:

$$q_0=505;$$

$$q_1=2305;$$

$$H=\frac{1}{48}(5\times505+2305)\frac{\overline{60}^2}{15}=24150 \text{ lbs.};$$

$$y_1=2.70 \text{ ft.}$$

CHAPTER XII

DETERMINING THE THICKNESS FOR ANY ARCH

LET CE (Fig. 37) be the line of resistance for any continuous loading q. At a distance x from the crown C, the vertical component of the thrust along the line of resistance is $V=\int_0^x q\cdot dx$, the horizontal component is H; hence the slope of the tangent to the resistance curve is

$$\tan\phi=\frac{dy}{dx}=\frac{V}{H}=\frac{1}{H}\int_0^x q\cdot dx.$$

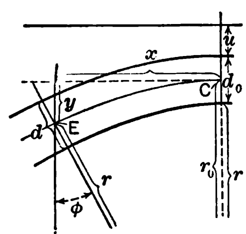

FIG. 37.—Determining the Thickness of Rib.

Differentiating this, we obtain,

$$\frac{d^2y}{dx^2}=\frac{q}{H}.$$

If r is the radius of curvature of the resistance line at the point E, then we also have

$$\frac{d^2y}{dx^2}=\frac{1}{r}\cdot\sec^3\phi;$$

consequently

$$H=q\cdot r\frac{1}{\sec^3\phi}=q\cdot r\cdot\cos^3\phi.$$

If the corresponding quantities at the crown are q_0 and r_0, then

$$H=q_0\cdot r_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

H attains its maximum value with the arch completely loaded. If p is a uniformly distributed load adopted as the equivalent

of the actual live load, and if g_0 is the dead load at the crown of the arch, then

$$H = (g_0 + p)r_0,$$

where r_0 denotes the radius of curvature at the crown of the resulting line of resistance. If we assume, with sufficient accuracy for the present investigation, that this resistance-line coincides with the axis of the arch, we may substitute for r_0 in the above formula the *radius of curvature of the arch-axis* at the crown.

If r' is the radius of curvature of the intrados, then

$$r' = r_0 - \frac{d_0}{2},$$

provided the rib is of uniform depth, d_0. As a rule, however, the depth of rib increases continuously from the crown toward the abutments. If the full load were the determining load for all sections, it would be necessary, in order to have the same total compression in every section, to make the thickness of rib $d = d_0 \cdot \sec \phi$. This rate of increasing depth, and sometimes an even greater variation, is always adopted for arches of larger spans, so that $r_0 > r' + \frac{d_0}{2}$. For the rate of increase of depth expressed by $d = d_0 \cdot \sec \phi$, we obtain $r_0 = r' + d_0$ and therefore $H = q_0 \cdot r_0 = q_0(r' + d_0)$.

Let s_0 = the intensity of stress in the crown section, assuming uniform distribution of pressure;

$A_0 = d_0 \cdot 1$ = the cross-section of the rib at the crown;

γ = the density of the material of the rib;

γ_1 = the density of the filling material;

u_0 = the depth of the spandrel filling above the crown of the rib;

p = the equivalent uniform value of the live load.

Then

$$H = d_0 \cdot s_0 = (\gamma \cdot d_0 + \gamma_1 \cdot u_0 + p)r_0 = (\gamma \cdot d_0 + \gamma_1 \cdot u_0 + p)(r' + d_0);$$

hence

$$d_0 = \frac{(\gamma_1 \cdot u_0 + p) \cdot r_0}{s_0 - \gamma r_0}, \quad \ldots \ldots \quad (82)$$

or

$$d_0 = \frac{(\gamma_1 \cdot u_0 + p) \cdot r'}{s_0 - \gamma r' - q_0}. \quad \ldots \ldots \quad (83)$$

If all units are given in terms of feet and pounds, then we may substitute for γ and γ_1 the densities of the respective materials, viz.:

Quarry stone		137 to 156
Ashlar masonry:	sandstone or limestone (medium)	137 to 150
	sandstone or limestone (heavy)	156 to 162
	granite	165
Brick		112
Concrete		137 to 150
Reinforced concrete		150 to 156

In using Eq. (83), the value of q_0 must first be estimated. For this purpose it is sufficient to assume the depth of rib by trial or by empirical formulæ.

In regard to the value to be adopted for the live load p, the following should be noted. The customary value for the weight of crowds of people, 80 to 100 pounds per square foot, is adequate only for foot-bridges. For vehicular traffic, a uniform live load should be substituted which would produce essentially the same effect as the axle-concentrations. A certain spreading of these loads through the spandrel filling or backing may be counted upon; so that, as the filling increases in depth, a smaller value for the equivalent load may be adopted. Also, as in the case of trusses, the equivalent uniform load may be reduced as the span increases.

The above considerations are expressed in the following empirical formulæ for the loads p to be assumed in different cases: (Units = pounds per square foot).

$$\left.\begin{array}{ll}
\textit{Highway bridges:} & \\
\quad\text{Very heavy vehicles,} & p=\left(100+\dfrac{12000}{l}\right)\dfrac{3+u_0}{3u_0} \\
\quad\text{Heavy vehicles,} & p=\left(100+\dfrac{6500}{l}\right)\dfrac{3+u_0}{3u_0} \\
\quad\text{Light vehicles,} & p=\left(100+\dfrac{2600}{l}\right)\dfrac{3+u_0}{3u_0} \\
\textit{Railway bridges:} & p=\left(1000+\dfrac{20000}{l}\right)\dfrac{3+u_0}{7+4u_0}
\end{array}\right\} \quad . \quad (84)$$

Here l is the loaded length of span (in feet) and u_0 the depth of filling at the crown (in feet).

These values for the live load may be used for the design of arches. Of course, to obtain more accurate results, the actual axle-weights of the wagons or road rollers should be used as concentrations, although allowance may be made for the spreading of the load by the filling and the material of the arch. In such computation, it is particularly important to consider the transverse width over which loads are spread. If u is the depth of filling or backing and b the width of tire or road roller, the wheel concentration G may be assumed to spread over a width

$$a=b+2u.$$

Since u increases from crown to springing, this would give a greater transverse spreading of wheel loads near the springing than at the crown. Nevertheless this variation may be disregarded and a mean depth of filling used for calculating the width a. At any rate, the assumed value of a for a wheel load cannot exceed the gauge of the wagon; and for an axle load in a railway, it cannot exceed about 13 feet. In accordance with the above, in designing an arch strip of unit width, the wheel loads will be treated as concentrations of the value $G \div a$.

In the preceding formulæ for the depth of rib d_0, it remains to select a value for the stress s_0. It would not be correct

to substitute the maximum allowable compressive stress of the material of the arch; for, aside from the fact that the assumed uniform distribution of pressure is not realized even at the crown in hingeless arches, the resistance-line for unsymmetrical loading deviates still more from the arch-axis at the other sections, so that extreme fiber stresses would then appear greatly in excess of the allowable intensities. We must therefore choose such a value for s_0 that, even with maximum deviation of the line of resistance, the extreme stresses should never exceed the safe working stress of the material.

1. Hingeless Arch

We will presuppose an arch conforming to the line of resistance (for a full-span load of $\frac{1}{2}p$) and assume that a loading of one-half of the span with the load p_1 yields the line of resistance departing farthest from the axis of the arch. We then find that aside from the moments at the ends, which receive extra stiffening, the greatest moments occur near the quarter-points of the span; and their value, by Eq. (49) (Chapter VII), is

$$M = \frac{9}{1024} p_1 l^2,$$

or, in round numbers, $M = \frac{1}{100} p_1 l^2$.

If H_1 is the horizontal thrust for loading on one-half the span, and if $d_1 = d_0 \cdot \sec\phi$ is the depth of rib at any section inclined at an angle ϕ, the maximum fiber compression at the section is

$$s = \frac{H_1 \cdot \sec\phi}{d_0 \cdot \sec\phi} + \frac{6}{100} \frac{p_1 l^2}{d_0^2} \cdot \cos^2\phi.$$

With the live load p over the full span and the dead load g_0 at the crown, we may write:

$$H = (g_0 + p) r_0 = d_0 \cdot s_0,$$

and, for half-span loading,

$$H_1 = g_0 \cdot r_0' + \tfrac{1}{2} p_1 \cdot r_0''.$$

If we assume the radii of curvature of the lines of resistance to be equal for the three cases of loading: dead load, live load and combined load, i.e., $r_0' = r_0'' = r_0$, and to equal the radius of the arch-axis at the crown, we obtain

$$H_1 = \frac{g_0 + \frac{1}{2} p_1}{g_0 + p} d_0 \cdot s_0.$$

Hence

$$s = \frac{g_0 + \frac{1}{2} p_1}{g_0 + p} \cdot s_0 + 0.06 \frac{p_1 l^2}{d_0^2} \cdot \cos^2 \phi$$

$$= \frac{g_0 + \frac{1}{2} p_1}{g_0 + p} \cdot s_0 + 0.06 \frac{p_1 l^2 \cdot \cos^2 \phi}{d_0 \cdot H} \cdot s_0.$$

Substituting the value of $H = (g_0 + p) r_0 = (g_0 + p) \dfrac{l^2}{8 f_0}$, where f_0 is the modified rise defined by Eq. (80), we obtain

$$s = \left(g_0 + \frac{1}{2} p_1 + 0.48 p_1 \frac{f_0}{d_0} \cos^2 \phi \right) \frac{s_0}{g_0 + p}.$$

With a slight rounding off of the numerical coefficient on the side of safety, we finally obtain

$$s_0 = \frac{g_0 + p}{g_0 + \frac{1}{2} p_1 \left(1 + \frac{f_0}{d_0} \right)} \cdot s. \quad . \quad . \quad . \quad . \quad . \quad (85)$$

It is thus necessary to estimate the value of d_0 as closely as possible in advance in order to calculate s_0; the latter is then substituted in (82) or (83) to determine the more correct value of d_0.

To avoid this roundabout procedure, it is preferable to obtain the value of d_0 directly from the equation

$$s = \frac{H_1}{d_0} + 0.06 \frac{p_1 l^2}{d_0^2} \cdot \cos^2 \phi.$$

Substituting

$$H_1=(g_0+\tfrac{1}{2}p_1)\cdot r_0, \quad \text{and} \quad g_0=d_0\gamma+u_0\gamma_1,$$

we obtain

$$d_0^2(s-\gamma r_0)=(u_0\gamma_1+\tfrac{1}{2}p_1)r_0\cdot d_0+0.06p_1l^2\cdot\cos^2\phi.$$

Introducing the abbreviation

$$w_2=u_0\gamma_1+\tfrac{1}{2}p_1, \quad \ldots\ldots \quad (86)$$

and with $l^2=8f_0r_0$, we finally obtain

$$d_0=\frac{1}{2}\,\frac{w_2\cdot r_0}{s-\gamma r_0}\left[1+\sqrt{1+\frac{2p_1f_0(s-\gamma r_0)}{w_2^2\cdot r_0}\cdot\cos^2\phi}\right]. \quad . \quad (87)$$

As an approximation, we may substitute the value

$$\cos^2\phi=\frac{l^2}{l^2+4f^2},$$

or, for comparatively flat arches, $\cos^2\phi=1$.

The following values may be taken for the maximum allowable compression s in the material of the arch, to be substituted in the above formulæ for calculating the thickness of arches:

	s (lbs. per sq. ft.)
Hard-burned brick in Portland cement mortar.	30,000 – 40,000
Quarry stone—medium hard stone in Portland cement mortar.	40,000 – 60,000
Coursed masonry—very hard stone.	60,000 – 80,000
Granite ashlar.	100,000 – 120,000
Plain concrete (1 : 5 to 1 : 3).	50,000 – 80,000

In arches of masonry or plain concrete it is not sufficient to keep the compressive stresses within a certain limit, but we must also take care that no excessive tensile stresses occur. If we consider merely the action of the loading apart from the effects of temperature, no tensile stresses at all should be

allowed in the arch-masonry, for otherwise the addition of temperature stresses would surely result in the low allowable stress being exceeded. We will therefore adhere to the rule that the line of resistance must remain within the middle third of the depth of the rib.

For the half-span load on the arch considered above, the maximum vertical distance of the line of resistance from the axis of the arch is approximately $h=\frac{1}{100}\frac{p_1 l^2}{H_1}$; and the radial distance is therefore $e=\frac{1}{100}\frac{p_1 l^2}{H_1}\cdot\cos\phi$. Since the line of resistance must not pass outside of the middle third of the section, we should have

$$d_1 \geqq 6e;$$

or, with $d_1=d_0\cdot\sec\phi$,

$$d_0 \geqq 0.06\frac{p_1 l^2}{H_1}\cos^2\phi.$$

Substituting $H_1=(g_0+\frac{1}{2}p_1)r_0$ and $l^2=8f_0 r_0$, we obtain

$$d_0 \geqq 0.48\frac{p_1\cdot f_0}{g_0+\frac{1}{2}p_1}\cos^2\phi. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (88)$$

With $g_0+\frac{1}{2}p_1=\gamma d_0+\gamma_1\cdot u_0+\frac{1}{2}p_1=\gamma d_0+w_2$, and rounding off the numerical coefficient to 0.5, there finally results

$$d_0 \geqq -\frac{w_2}{2\gamma}+\sqrt{\frac{w_2{}^2}{4\gamma^2}+\frac{p_1 f_0}{2\gamma}\cos^2\phi}. \quad . \quad . \quad . \quad . \quad . \qquad (89)$$

This relation fixes the minimum crown-depth of a hingeless arch in which no tensile stresses are produced by the external loading.

In long-span arches or in those of long radius and large dead weights, the compressive stress is usually the determining factor, and the required depth of rib is given by Eq. (87). Smaller arches having a relatively greater live load may be found by Eq. (89) to require a greater thickness in order to

avoid tensile stresses. In such arches the compressive strength of the material is not fully utilized.

It should be kept in mind that the above formulæ for depth of rib have been *derived under the assumption of an arch-curve conforming to the mean position of the line of resistance.* It has also been assumed that the radial sections increase in depth toward the springing so that their vertical projection remains constant. If these conditions are not fulfilled, the thickness of the rib must be correspondingly increased. The effect of temperature will require a special stiffening of the rib at the ends. The necessity for such increased thickness will be determined by the results of the more exact investigation of the stresses.

EXAMPLE. Arch bridge for very heavy highway traffic. Span 100 ft., rise 16 ft. Built of cut masonry having a compressive strength in test cubes of about 5700 to 7000 lbs. per sq.in. Density of the stone, $\gamma=150$; of the earth filling, $\gamma_1=110$. Depth of filling above crown, $u_0=1.6$ ft.; roadway horizontal.

We assume the live load by Eq. (84): for full-span load,

$$p=\left(100+\frac{12000}{100}\right)\frac{4.6}{4.8}=\text{round } 210\ \frac{\text{lbs.}}{\text{ft.}^2}$$

for half-span load,

$$p_1=\left(100+\frac{12000}{50}\right)\frac{4.6}{4.8}=\text{round } 330\ \frac{\text{lbs.}}{\text{ft.}^2}$$

We first calculate the proper curve for the arch by Eqs. (71) to (74), with the thickness at the crown assumed provisionally as $d_0=2.5$ ft. We obtain:

$$q_0=2.5\times150+1.6\times110+\tfrac{1}{2}\times210=656 \text{ lbs. per sq.ft.}$$

$$q''=\frac{8}{3}\left(\frac{16}{100}\right)^2 2.5(150-55)=16 \text{ lbs. per sq.ft.};$$

$$H=672\frac{\overline{100}^2}{16\times16}\left[1+\sqrt{1+\frac{2}{3}\frac{656}{\overline{672}^2}\cdot110\times16}\right]=69{,}400 \text{ lbs. per ft.}$$

$$y=(656+0.09 31x^2)\frac{x^2}{138800}=(0.00473+0.000000671x^2)x^2;$$

$$r_0=\frac{69400}{656}=106 \text{ ft.};$$

$$f_0=\frac{\overline{100}^2}{8\times106}=11.8 \text{ ft.}$$

For the material described above, we may assume a working compressive stress of $s=425$ lbs. per sq.in. Substituting $w_2=1.6\times110+\frac{1}{2}\times330=341$ and $\cos^2\phi=\frac{\overline{100}^2}{\overline{100}^2+4(16)^2}=0.9$, we obtain the crown thickness by Eq. (87):

$$d_0=\frac{1}{2}\,\frac{341\times106}{425\times144-150\times106}\left[1+\sqrt{1+\frac{2\times330\times11.8(61200-15900)\times0.9}{(341)^2\cdot106}}\right]$$
$$=2.47 \text{ ft.}$$

If Eq. (89) is used, we obtain the following value for the crown thickness:

$$d_0=-\frac{1}{2}\,\frac{341}{150}+\sqrt{\left(\frac{341}{300}\right)^2+\frac{330\times11.8\times0.9}{2\times150}}=2.46 \text{ ft.}$$

The assumed value of the crown thickness ($d_0=2.5$) is therefore safe by a narrow margin; nevertheless slight tensile stresses must be expected under unfavorable conditions of loading. It thus appears that in masonry arches, as in the above example, the use of stronger stone brings no advantage, since the increased compressive strength cannot be utilized; the necessity of avoiding tension determines the thickness.

Since the action of temperature changes will produce additional stresses, it appears advisable to increase the above thickness of the arch; it is therefore made 2.8 ft. at the crown and 4.6 ft. at the springing.

2. Three-hinged Arches

In an arch having hinges at the crown and ends of the span, Eq. (82) may be employed to indicate the proper depth at the crown; but in this case we may substitute for s_0 a value nearer the full allowable compressive stress of the material. At the same time the thickness must be increased at the critical sections in the haunches to keep the extreme fiber stresses within safe limits and, in the absence of reinforcement, to prevent the appearance of tensile stresses. If the arch-curve conforms to the mean position of the resistance-line (for a full-span load of $\frac{1}{2}p$), the bending moment at the quarter-points for a load covering the half-span will be (page 20)

$$M=\tfrac{1}{64}p_1l^2,$$

and the corresponding fiber stresses will be

$$s=\frac{H_1 \sec \phi}{d_1} \pm \frac{6}{64}\frac{p_1 l^2}{d_1^2},$$

where d_1 is the depth of section (Fig. 38). Substituting

$$H_1=(g_0+\tfrac{1}{2}p_1)r_0 \quad \text{and} \quad l^2=8f_0r_0 \ ,$$

we obtain

$$d_1 \cos \phi=\frac{1}{2}\frac{(g_0+\frac{1}{2}p_1)}{s}r_0\left[1+\sqrt{1+\frac{3p_1f_0s}{(g_0+\frac{1}{2}p_1)^2r_0}\cdot\cos^2\phi}\right]. \quad (90)$$

where s is the safe limit of compression.

If no tensile stresses are permissible, we must have

$$\frac{H_1 \sec \phi}{d_1}-\frac{6}{64}\frac{p_1 l^2}{d_1^2} \geqq 0$$

or

$$d_1 \cos \phi \geqq \frac{3}{4}\frac{p_1 f_0}{g_0+\frac{1}{2}p_1}\cdot\cos^2\phi. \quad . \quad . \quad . \quad . \quad (91)$$

At the crown of three-hinged arches the thickness is generally made less than $d_1 \cos \phi$, and at the springing it is made less

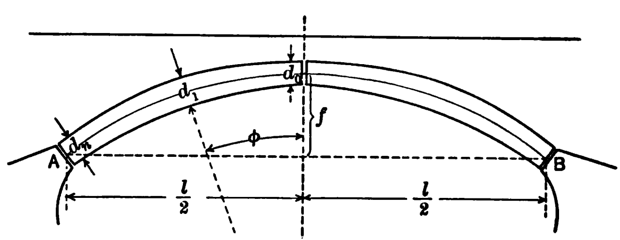

FIG. 38.—Thickness of Rib for Three-hinged Arch.

than d_1; so that the greatest thickness occurs at the middle of the haunches (Fig. 38). The assumption $d_1=d_0 \sec \phi$ used in deriving the equations for the arch-curve is therefore unfulfilled, and the corresponding Eqs. (75) or (79) for r_0 is no longer accurate. Nevertheless that value should be used in Eq. (90) or (91) to calculate the approximate value of the thickness of the arch for a preliminary design. This preliminary design

gives corrected dead weights with which H and H_1 may be recalculated, the corresponding lines of resistance drawn and the curvature and thickness of the arch revised.

EXAMPLE. (The Iller Bridge at Kempten, Bavaria).—Railway bridge, three-hinged arch of plain concrete. Span 164 ft., rise 28.5 ft.; earth covering 2.6 ft.; spandrels filled with concrete to the level of the crown. For this case, the live load to assume according to Eq. (84) would be:

For full-span load,

$$p=\left(1000+\frac{20000}{164}\right)\frac{3+2.6}{7+10.4}=360 \text{ lbs. per sq.ft.;}$$

for the half-span load,

$$p_1=\left(1000+\frac{20000}{82}\right)\frac{3+2.6}{7+10.4}=400 \text{ lbs. per sq.ft.}$$

Instead we will use the heavier loads required by the Bavarian specifications, which would give: for full-span load, $p=400$ lbs. per sq.ft.; for half-span load, $p_1=525$ lbs. per sq.ft.

Assuming a crown thickness of 4.4 ft., and densities for concrete and filling of $\gamma=140$ and $\gamma_1=110$, respectively, we find, at the crown,

$$q_0=4.4(140)+2.6(110)+\frac{400}{2}=1102 \text{ lbs. per sq.ft.;}$$

at the springing,

$$q_1=33.5(140)+2.6(110)+\frac{400}{2}=5176 \text{ lbs. per sq.ft.}$$

Substituting these values in Eq. (79), we obtain

$$r_0=\frac{1}{6}\left(5+\frac{5176}{1102}\right)\frac{\overline{164}^2}{8\times28.5}=190 \text{ ft.}$$

$$f_0=\frac{\overline{164}^2}{8\times190}=17.7 \text{ ft.}$$

Assuming a maximum compressive stress of $s=425$ lbs. per sq.in., Eq. (90) gives

$$d_1\cos\phi=\frac{1}{2}\,\frac{(902+262.5)190}{425\times144}\left[1+\sqrt{1+\frac{3\times525\times17.7\times425\times144}{(1164.5)^2\cdot190}}\cdot0.97\right]=6.7 \text{ ft.}$$

Hence $$d_1=6.8 \text{ ft.}$$

Eq. (91) would yield

$$d_1 \cos \phi > \frac{3}{4} \cdot \frac{525 \times 17.7}{1164.5} \cdot 0.97 = 5.8 \text{ ft.},$$

or

$$d_1 \overline{\overline{>}} 5.9 \text{ ft.}$$

The horizontal thrust for a full-span load will be approximately

$$H = (4.4 \times 140 + 2.6 \times 110 + 400) \cdot 190$$
$$= 1302 \times 190 = 247{,}500 \text{ lbs. per ft. (of width)}$$

and the fiber stress at the crown will be

$$s_0 = \frac{247500}{4.4 \times 144} = 390 \text{ lbs. per sq.in.}$$

The vertical reaction at the ends under the full-span load will be approximately

$$\left[1302 + \frac{1}{3}(q_1 - q_0)\right]\frac{l}{2} = 218{,}000 \text{ lbs. per ft.}$$

Hence the resultant end-reaction will be

$$\sqrt{\overline{247500}^2 + \overline{218000}^2} = 330000 \text{ lbs. per ft.}$$

Adopting the same fiber stress as at the crown, $s_0 = 390$ lbs. per sq.in., we obtain the thickness at the springing

$$d_2 = \frac{330000}{390 \times 144} = 5.9 \text{ ft.}$$

The dimensions actually provided are $d_0 = 4.4$ ft., $d_1 = 6.5$ ft., $d_2 = 6.0$ ft. The exact design of the arch yields a maximum fiber stress of 500 lbs. per sq.in. which is a satisfactory check on the above approximate determination of dimensions.

CHAPTER XIII

PROPORTIONING REINFORCED CONCRETE ARCHES

In the last chapter, formulæ were developed for the thickness of masonry and concrete arches to satisfy the requirements: first, that the allowable compressive stress should not be exceeded; second, that no tensile stresses should occur in the masonry. The second requirement, under certain conditions, may call for a greater thickness than is necessary merely to limit the compression; this occurs with large ratios of live to dead load, large rise-ratios, or with material having a high allowable unit stress in compression. By employing a material adapted to resist tensile stresses, such as reinforced concrete or combination arches of steel and concrete, the above difficulty is eliminated. Furthermore, as the embedded steel also shares in resisting the compression, reinforced concrete arches do not require as great thickness as plain concrete arches.

Let the area of steel per unit width of rib be given by $A_s = a_0 \cdot d_0$, where d_0 is the crown thickness. Here a_0 is the ratio (not percentage) of reinforcement at the crown. The reinforcement is assumed symmetrical about the arch-axis, so that the steel and the concrete at any section will have a common gravity axis.

The cross-section at the crown reduced to its equivalent in concrete amounts to $d_0(1+na_0)$. Let s_0 denote the stress in the concrete at the crown under central action of the resultant. The horizontal thrust produced by a load p over the full span amounts to $H=(\gamma d_0+g'+p)\cdot r_0$, where g' is the dead load above the crown of the arch. Then

$$(\gamma d_0+g'+p)\cdot r_0=d_0(1+na_0)\cdot s_0.$$

Introducing the symbol

$$\gamma' = \frac{1}{1+na_0} \cdot \gamma,$$

we obtain

$$d_0' = d_0(1+na_0) = \frac{(g'+p)r_0}{s_0-\gamma' r_0}. \quad . \quad . \quad . \quad . \quad (92)$$

This equation is identical with Eq. (82) for the crown thickness of a plain concrete arch of density γ'.

If the thickness d_0 is chosen to prevent excessive compression in the concrete at the most severely stressed section, we have to start with the relation

$$f_c = \frac{H_1 \cdot \sec\phi}{A} + \frac{M}{Z} \text{ (where } Z \text{ is the section modulus).}$$

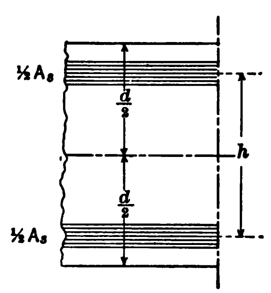

FIG. 39.—Symmetrical Reinforcement.

We assume that the tensile stresses in the critical section are so low (under 150 lbs. per sq.in.) that the full section of the concrete may be considered acting and may be designed according to Phase I (page 9). Accordingly, if d is the depth of the section, a_0 the reinforcement ratio, and h the distance between the centers of gravity of the two layers of reinforcement (Fig. 39), we have

$$A = d(1+na_0);$$

$$I = \frac{1}{12}d^3 + na_0 d\frac{h^2}{4};$$

$$\therefore \quad Z = \frac{1}{6}d^2 + na_0\frac{h^2}{2}.$$

Assuming $h = 0.85d$, we may write approximately

$$Z = \tfrac{1}{6}d^2(1+2na_0) = \tfrac{1}{6}d^2(1+na_0)^2,$$

so that we obtain

$$f_c = \frac{H_1 \cdot \sec \phi}{d(1+na_0)} + \frac{6M}{d^2(1+na_0)^2}. \quad . \quad . \quad . \quad . \quad (93)$$

1. Hingeless Arch

Let the thickness at the crown be denoted by d_0. Assume that the thickness increases gradually toward the abutments, about as the secant of the inclination of the sections, so that we may write $d = d_0 \cdot \sec \phi$. We also have

$$H_1 = (\gamma d_0 + g' + \tfrac{1}{2} p_1) r_0 = (\gamma d_0 + w_2) r_0.$$

Assuming that the axis of the arch is conformed to the mean line of resistance, the greatest bending moment under a half-span load will be approximately

$$M = \frac{1}{100} p_1 l^2.$$

Solving Eq. (93), we obtain the following expression for the crown thickness analogous to Eq. (87) for the non-reinforced arch:

$$d_0' = d_0(1+na_0) = \frac{1}{2} \frac{w_2 \cdot r_0}{f_c - \gamma' r_0} \left[1 + \sqrt{1 + \frac{2 p_1 f_0 (f_c - \gamma' r_0) \cos^2 \phi}{w_2{}^2 \cdot r_0}} \right], \quad (94)$$

where r_0, f_0, w_2 are defined by previous Eqs. (79), (80) and (86), and $\gamma' = \frac{1}{1+na_0} \gamma$.

From d_0', assuming the ratio of reinforcement, we can calculate the crown thickness

$$d_0 = \frac{1}{1+na_0} \cdot d_0', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

or, assuming d_0, we can calculate the necessary ratio of reinforcement a_0.

The cross-section will then be adequately reinforced for the compressive stresses. It remains to be tested whether the reinforcement thus calculated is sufficient also to take care of the tensile stresses. For the purpose of the preliminary proportioning of the sections we have to resort to a rough approximate method; the subsequent exact investigation of the maximum stresses at each section will determine whether any changes must be made in the thickness and reinforcement of the arch.

With the entire section of the concrete effective, the two extreme fiber stresses will be:

compression, $$f_c = \frac{6M}{d_0'^2} \cdot \cos^2 \phi + \frac{H_1}{d_0'};$$

tension, $$f_{ct} = \frac{6M}{d_0'^2} \cdot \cos^2 \phi - \frac{H_1}{d_0'}.$$

The tension zone of the concrete will therefore have the height

$$x = \frac{f_{ct}}{f_c + f_{ct}} \cdot d = \left(\frac{1}{2} - \frac{H_1 \cdot d_0'}{12M} \cdot \sec^2 \phi \right) \cdot d$$
$$= \left[\frac{1}{2} - \frac{100(\gamma d_0 + w_2) r_0 d_0'}{12 p_1 \cdot 8 r_0 f_0} \cdot \sec^2 \phi \right] d.$$

Hence, approximately,

$$x = \left[\frac{1}{2} - \frac{(\gamma d_0 + w_2)}{p_1} \cdot \frac{d_0'}{f_0} \cdot \sec^2 \phi \right] d.$$

The total tension to be provided for will then be

$$T = \frac{1}{2} f_{ct} \cdot x = \left[\frac{1}{2} - \frac{\gamma d_0 + w_2}{p_1} \cdot \frac{d_0'}{f_0} \cdot \sec^2 \phi \right]^2 \cdot \frac{p_1 f_0}{2 d_0'^2} \cdot r_0 \cdot d \cdot \cos^2 \phi.$$

If we specify that the concrete itself cannot take any tensile stress, so that the entire resistance to tension must be supplied by the steel, and if, with slight error, we assume that the

tension in the steel $=T$, then, with f_s as the stress in the steel, we have

$$T=\tfrac{1}{2}A_s\cdot f_s=\tfrac{1}{2}a_0\cdot d\cdot f_s.$$

Equating this to the preceding expression for T, and solving for a_0, we obtain

$$a_0=\left[\frac{1}{2}-\frac{(\gamma d_0+w_2)d_0'}{p_1\cdot f_0}\cdot\sec^2\phi\right]^2\frac{p_1 f_0 r_0}{d_0'^2\cdot f_s}\cdot\cos^2\phi. \quad . \quad . \quad (96)$$

As a rule, if the rise of the arch is not too high or the live load too heavy, this second condition (96) will yield smaller values for the required reinforcement than the first condition Eq. (95).

In the above, the stresses produced by temperature and loading in the sections near the abutments have not been considered. These will require, in most cases, a greater thickness than that indicated by the rule of $d=d_0\cdot\sec\phi$, or else an increase in the reinforcement.

2. Three-hinged Arch

Formulæ analogous to those derived above for the hingeless arch may be established for the three-hinged arch. We again assume that the axis of the arch has been made to conform to the mean line of resistance, and proceed to find the thickness d_1 at the middle of the haunches. As in the case of the hingeless arch, we obtain an expression identical with that for a non-reinforced arch (Eq. (90)):

$$d_1'\cos\phi=d_1(1+na_1)\cos\phi$$
$$=\frac{1}{2}\frac{(g_0+\frac{1}{2}p_1)r_0}{f_c}\left[1+\sqrt{1+\frac{3p_1 f_1 f_c}{(g_0+\frac{1}{2}p_1)^2 r_0}\cdot\cos^2\phi}\right]. \quad (97)$$

Here $g_0(=\gamma\cdot d_0+g')$ denotes the dead load at the crown.

Since no additional stresses can be produced in three-hinged arches by temperature or yielding abutments, it will be proper to adopt a higher unit stress f_c than in the hingeless arch.

The minimum reinforcement ($A_s = a_1 d_1$) required to take care of the tensile stresses is found in the same manner as in the hingeless arch, giving

$$a_1 = \left[\frac{1}{2} - \frac{2}{3}\frac{g_0 + \frac{1}{2}p_1}{p_1} \cdot \frac{d_1'}{f_0} \cdot \sec \phi\right]^2 \cdot \frac{3}{2}\frac{p_1}{d_1'^2}\frac{f_0 r_0}{f_s}. \quad . \quad . \quad (98)$$

As before, this value will usually be exceeded by the amount of reinforcement required for the compressive stresses as determined by the ratio $d_1' : d_1$.

3. Rigid Reinforcement Carrying Part of the Dead Load. (The Melan System.)

If the arch-curve conforms to the correct line of pressure and the live load is not too great, only small tensile stresses will appear. Furthermore, since the maximum intensity of compression in the steel cannot exceed $n = 15$ times the value for concrete, or about 6000 to 7500 lbs. per sq.in., it is evident that *the full strength of the steel cannot be utilized in the ordinary arches of reinforced concrete.* This waste of strength may be avoided, however, by using self-supporting reinforcement (as in the Melan system), and hanging the forms on these steel reinforcing ribs; a portion of the dead weight of the structure is thus transferred directly to the steel, so that the latter receives a certain initial stress before the concrete is stressed at all.

Let the arrangement of the centering be such that it carries only a fraction k of the total weight of the concrete arch, the balance $(1-k)$ being transferred directly to the steel reinforcing ribs by suspending the forms from the latter. We may then apply Eq. (94) for the hingeless arch or Eq. (97) for the three-hinged arch upon substituting in this case

$$\gamma' = \frac{k}{1 + na_0}\gamma \quad \text{and} \quad g_0' = k \cdot \gamma \cdot d_0 + g',$$

or, in general, substituting $k\gamma$ for γ.

The required strength of the steel ribs is then estimated as follows:

The weight of the construction suspended from the steel ribs produces a compressive stress f_s'; if we assume that the line of resistance for this loading also coincides with the arch-axis, we have

$$f_s' = \frac{(1-k)\gamma \cdot d_0 \cdot r_0 \cdot \sec\phi}{A_s} = \frac{(1-k)\cdot\gamma\cdot r_0}{a_0}.$$

To this must be added the compression given to the steel as part of the composite arch. This cannot exceed n times the concrete stress f_c, or $f_s'' = n \cdot f_c$. Consequently the total stress in the steel will be

$$f_s = f_s' + f_s'' = \frac{(1-k)\gamma\cdot r_0}{a_0} + nf_c;$$

hence

$$a_0 = \frac{(1-k)\gamma\cdot r_0}{f_s - n\cdot f_c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

Similarly, in the three-hinged arch, the area of reinforcement $A_s = a_1 d_1$ at the thickest section d_1 is given by

$$a_1 = \frac{(g_0 - g_0')r_0 \cdot \sec\phi}{d_1(f_s - n\cdot f_c)}. \quad . \quad . \quad . \quad . \quad . \quad (100)$$

The steel stress f_s in these formulæ should not be taken greater than about 10,000 lbs. per sq.in., as the initial stress $(f_s - n\cdot f_c)$ must be kept down to a low value on account of the danger of buckling of the steel ribs and the existence of bending stresses in addition to the axial stresses considered above.

1. Example. (Fig. 40.)*—Highway bridge. Hingeless, reinforced concrete arch. Span $l = 113$ ft., rise $f = 12.75$ ft. Depth of stone filling above the crown $= 1$ ft. Relieving arches in the spandrels.

The live load is assumed at the values $p = 200$ lbs. per sq.ft. for full-span loading, $p_1 = 270$ lbs. per sq.ft. for half-span loading. The crown

* Taken from *Melan* and *Kluge*: "Einige neuere Brückenausführungen nach Bauweise Melan." Second Edition, Berlin, 1911. Wm. Ernst & Son.

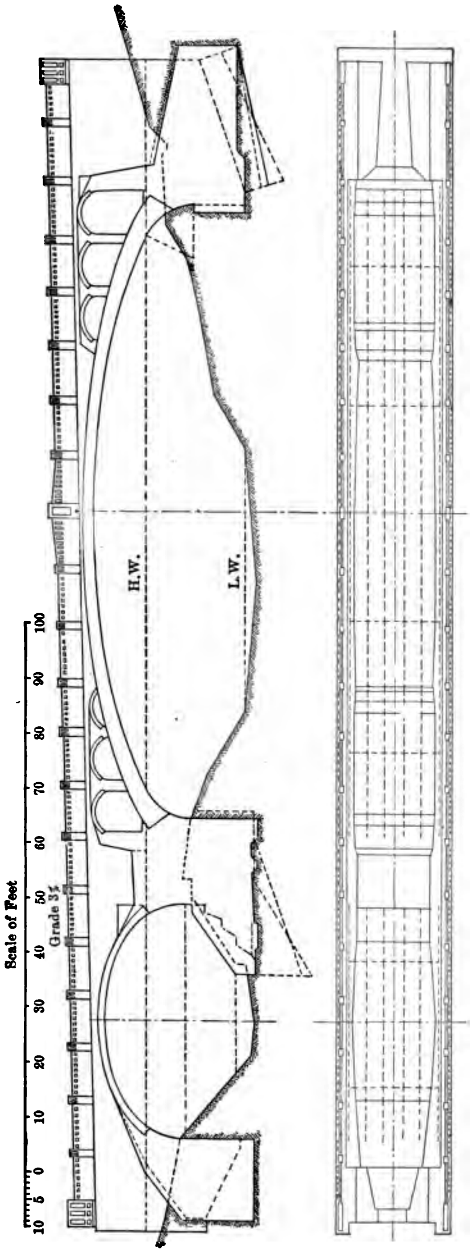

FIG. 40.—Highway Bridge. Example of the Melan Type of Arch.

thickness of the arch is assumed at a trial value of $d_0 = 2$ ft. From the preliminary layout, the total load at the crown is found to be

$$q_0 = \gamma d_0 + g' + \tfrac{1}{2}p = 156 \times 2 + 110 \times 1 + 100 = 522 \text{ lbs. per sq.ft.};$$

at the ends,

$$q_1 = \gamma d_1 + g'_1 + \tfrac{1}{2}p = 156 \times 3.3 + 505 + 100 = 1120 \text{ lbs. per sq.ft.}$$

Hence, by Eq. (79), the radius of curvature at the crown is

$$r_0 = \frac{1}{6}\left(5 + \frac{1120}{522}\right)\frac{\overline{113}^2}{8 \times 12.75} = 150 \text{ ft.};$$

and, by Eq. (80), the reduced arch-rise is

$$f_0 = \frac{\overline{113}^2}{8 \times 150} = 10.6 \text{ ft.}$$

We also have

$$w_2 = g' + \tfrac{1}{2}p_1 = 245 \text{ lbs. per sq.ft.},$$

$$\cos^2 \phi = 0.95.$$

Assuming a trial value of 2 per cent reinforcement ($a_0 = .02$),

$$\gamma' = \frac{\gamma}{1 + 15 \times .02} = \frac{156}{1.30} = 120,$$

$$\gamma' r_0 = 120 \times 150 = 18{,}000.$$

Specifying a compressive stress in the concrete of $f_c = 425$ lbs. per sq.in. = 61,200 lbs. per sq.ft., Eq. (94) gives

$$d_0' = \frac{1}{2}\,\frac{245 \times 150}{61200 - 18000}\left[1 + \sqrt{1 + \frac{2 \times 270 \times 10.6(43200)0.95}{\overline{245}^2 \times 150}}\right] = 2.64 \text{ ft.}$$

Retaining the trial value of the crown thickness, $d_0 = 2.0$ ft., we find the coefficient of reinforcement

$$a_0 = \frac{1}{15}\left(\frac{2.64}{2.0} - 1\right) = 2.1 \text{ per cent.}$$

Hence the required sectional area of reinforcement for 1 ft. width of rib is

$$A_s = 2.1 \text{ per cent of } 2.0 \text{ sq.ft.} = 6.1 \text{ sq.in.}$$

To provide for tensile stresses at the haunches, assuming the allowable tension in the steel $f_s = 10{,}000$ lbs. per sq.in., the required reinforcement is found by Eq. (96):

$$a_0 = \left[\frac{1}{2} - \frac{557 \times 2.64}{270 \times 10.6 \times 0.95}\right]^2 \frac{270 \times 10.6 \times 150 \times 0.95}{\overline{2.64}^2 \times 10000 \times 144} = 0,$$

i.e., there are no tensile stresses in this arch with the dimensions assumed above (except possibly at the springing), so that the compressive stresses alone govern the depth of rib and amount of reinforcement.

If the arch is built with *rigid* reinforcement (Melan type) from which the forms are suspended in such manner that it sustains about one-third of the weight of the arch, we find that less reinforcement can be used with the same depth of rib as before. With $(1-k) = \frac{1}{3}$ and $f_s = 10{,}000$ lbs. per sq.in., a minimum value for the required reinforcement is given by Eq. (99):

$$\text{Minimum } a_0 = \frac{1}{3} \cdot \frac{156 \times 150}{144(10000 - 15 \times 425)} = 1.5 \text{ per cent.}$$

We will try $a_0 = 1.6$ per cent, so that

$$\gamma' = \frac{2}{3} \cdot \frac{156}{1 + 15 \times 0.016} = 84 \text{ lbs. per cu.ft.};$$

$$\gamma' r_0 = 84 \times 150 = 12{,}600 \text{ lbs. per sq.ft.};$$

$$f_c - \gamma' r_0 = 61{,}200 - 12{,}600 = 48{,}600 \text{ lbs. per sq.ft.}$$

Substituting the last value in Eq. (94), we obtain

$$d_0' = \frac{1}{2} \cdot \frac{245 \times 150}{48600}\left[1 + \sqrt{1 + \frac{2 \times 270 \times 10.6 \times 48600 \times 0.95}{\overline{245}^2 \times 150}}\right] = 2.46 \text{ ft}$$

Keeping the thickness $d_0 = 2.0$ ft., we obtain

$$a_0 = \frac{1}{15}\left(\frac{2.46}{2.0} - 1\right) = 1.53 \text{ per cent};$$

and the required area of reinforcement for 1 ft. width of arch is

$$A_s = 1.53 \times 2.0 \times 144 = 4.4 \text{ sq.in.},$$

or a saving of 30 per cent over the first design.

It is thus apparent that the scheme of using a rigid, self-supporting reinforcement, to which part of the weight of the structure is initially transferred

by suspending the falsework therefrom, results in a marked saving in the amount of the reinforcement through the fuller utilization of its strength.

The above bridge was actually built with a crown thickness of 1.8 ft.; and with latticed reinforcing arches, spaced 3.3 ft. apart, having flange sections of 14.6 sq.ins., or 4.4 sq.ins. per ft. of width as determined above. The analysis of the stresses indicated a maximum compression in the concrete of 500 lbs. per sq.in.

2. EXAMPLE. (Design for a bridge in Switzerland.)—Three-hinged arch. Span 200 ft., rise 40 ft. The roadway is carried on plate girders resting on reinforced concrete piers. A street railway is to be carried over the bridge. Accordingly the following live loads are assumed:

for full-span loading, $p=270$ lbs. per sq.ft.
for half-span loading, $p_1=300$ lbs. per sq.ft.

For trial values, we take the thickness at the crown as 2.6 ft. and the depth of filling as 1.0 ft.; so that the dead load at the crown is

$$g_0=2.6\times 156+1.0\times 110=515 \text{ lbs. per sq.ft.}$$

With the results of a preliminary estimate of the dead loads and with a load $\frac{1}{2}p$ over the whole span, the resistance curve was determined for the arch, and the radius of curvature at the crown was found to be $r_0=162$ ft.

The compressive resistance of the concrete is assumed at $f_c=570$ lbs. per sq.in., with $n=10$. (Swiss specifications.) The allowable steel stress is $f_s=14{,}000$ lbs. per sq.in.

One-half the weight of the arch was suspended from the self-supporting steel arches used as reinforcement. Consequently $g_0'=g_0-203=312$ lbs. per sq.ft.

Substituting

$$f_0=\frac{l^2}{8r_0}=30.8;$$

$$\cos^2\phi=0.864;$$

$$\cos\phi=0.93;$$

Eq. (97) gives

$$d_1'\cos\phi=\frac{1}{2}\cdot\frac{462\times 162}{570\times 144}\left[1+\sqrt{1+\frac{3\times 300\times 30.8\times 570\times 144\times 0.864}{\overline{462}^2\times 162}}\right]=3.9 \text{ ft.}$$

Hence

$$d_1'=(1+na_1)d_1=\frac{3.9}{0.93}=4.2 \text{ ft.}$$

Choosing $d_1 = 3.9$ ft.,

we find $a_1 = \frac{1}{10}\left(\frac{d_1'}{d_1} - 1\right) = 0.75$ per cent.

In consideration of the steel stresses, however, we will make

$$a_1 = 1.16 \text{ per cent;}$$

hence Eq. (100) may be written

$$\frac{203 \times 162}{3.9(f_s - 10 \times 570) \times 144 \times 0.93} = 0.0116,$$

giving the maximum compression in the steel

$$f_s = 11,100 \text{ lbs. per sq.in.}$$

The maximum tension in the steel is found approximately by substituting g_0' in Eq. (98), giving

$$f_s = 9250 \text{ lbs. per sq.in.}$$

If the steel arches were made lighter, corresponding to the percentage of reinforcement (0.75) determined above, the steel stresses given by Eqs. (98) and (100) would amount to about 14,000 lbs. per sq.in. for both tension and compression. It does not appear advisable to allow the compression to attain this value, since the corresponding initial stress from the suspended construction would be 14,000 − 5700 = 8300 lbs. per sq.in., a value at which the danger of buckling is too great.

The adopted proportions, $d_1 = 3.9$ ft. and $a_1 = 1.16$ per cent, i.e., $A_s = 6.5$ sq.in. per ft. of width, may accordingly be considered proper and used as a basis for the further study of the design.

CHAPTER XIV

CALCULATING THE STRESSES IN AN ARCH

ANY arch of masonry or reinforced concrete proportioned tentatively according to the foregoing suggestions or by empirical rules should undergo a more rigorous investigation of the stresses according to the theory of the elastic arch (Chapters I to IX) in order to make sure that the stresses are within the prescribed limits or to show where the sections require strengthening.

Such investigation should include the effects of dead load, live load and the secondary stresses producible by variation of temperature or other disturbing influences.

With respect to the live load, it is quite sufficient in the case of smaller arches or in less exacting designs to consider only two cases of loading: full load, and load on one-half of the span, thus obtaining the maximum stresses at the crown and approximately at the haunches and springing. The stresses at any section should always be determined from the calculated values of the bending moment and axial thrust and not merely by drawing the line of resistance, since the accuracy of the graphic construction of the latter is insufficient for the determination of the stresses.

In the more thorough investigation of arches carrying heavier live loads, the stresses should be found for a series of sections with the most unfavorable condition of loading assumed for each. For this purpose the method of influence-lines can be applied with advantage, affording sufficient accuracy even with a purely graphic procedure; furthermore, without materially increasing the labor, this method enables the live load to be treated as a train of concentrations instead of replacing it by an equivalent uniform load.

The influence-lines for the bending moments and axial thrusts are easily determined in the case of a statically determinate *three-hinged arch* (Chapter III). The *hingeless arch* of masonry or concrete, according to the modern view, should be treated as an elastic rib with fixed ends, so that the methods of Chapter IV may be applied.

The *calculation of the stresses* in an arch is best performed separately: 1, for the dead loads; 2, for the live loads, assuming for each section the two severest conditions of loading producing the maximum positive and negative moments about the gravity axis; and 3, for the effect of temperature.

The dead load stresses may be found either analytically or by the use of influence lines. An exclusively analytical treatment for the investigation of the live load stresses is too laborious to carry out unless but two conditions of loading (half-span and full-span) are considered.

From the axial thrusts and moments, the fiber stresses in the sections are calculated in the well-known manner (Chapter II). These are arranged in tabular form and, by proper summation, the maximum stresses producible by dead load, live load and temperature are found.

The stresses in a reinforced concrete arch are generally calculated on the basis of Phase I, i.e., on the assumption that the full cross-section of the concrete is effective with a uniform coefficient of elasticity. For those cross-sections in which this method indicates tension in the concrete exceeding about 150 lbs. per sq.in., the stresses must be corrected by applying the method of Phase II. For this purpose we use Eq. (7) with $m=0$ or $m=0.4$. The assumption $m=0$ yields somewhat excessive values for the compression in the concrete and the tension in the steel.

For further information, the reader is referred to the appendix, which gives examples of the design and investigation of the stresses in arch bridges.

The *reinforcement* in an arch is usually disposed symmetrically about the axis and with uniform section throughout the arch. But if the arch has large bending moments with relatively

small axial thrusts, as is the case in arches of large rise-ratio, large ratio of live to dead load or, particularly when the arch does not conform to the mean line of resistance, a variation in the reinforcement becomes necessary. In such case we may use an approximate, direct method for proportioning the reinforcement; namely, after deducting the small stress due to the axial thrust from the total allowable compression in the concrete, we determine the necessary steel areas from the formulæ for pure bending in doubly reinforced beams.

If M_1 and M_2, the extreme values of the bending moment about the axis of any section, have opposite signs, and the axial thrust is small, a double reinforcement will be required. Substituting

$$\begin{cases} A_s' = a' \cdot d = \text{section of steel on tension side,} \\ A_s'' = a'' \cdot d = \text{section of steel on compression side,} \end{cases}$$

where a' and a'' are the respective ratios of reinforcement, and assuming that the steel is arranged in layers at distances of $0.1d$ from the upper and lower faces of the concrete, respectively, we find, with $n = E_s \div E_c = 15$, the following value for the moment of inertia of the effective section per unit width of rib:

$$I = 5[(0.9-x)(2.7-x)a' + (x-0.1)(x-0.3)a''] \cdot d^3 = i \cdot d^3,$$

where

$$x = -15(a'+a'') + \sqrt{\overline{15}^2(a'+a'')^2 + 30(0.9a' + 0.1a'')}$$

fixes the distance of the neutral axis from the extreme compression fiber as a fraction of the total depth of rib.

If f_c and f_s are the working stresses for concrete (in compression) and steel (in tension), the resisting moment of the section is

$$M = \frac{i}{x} \cdot d^2 \cdot f_c = m \cdot d^2 \cdot f_c,$$

or

$$M = \frac{i}{0.9-x} \cdot d^2 \cdot \frac{f_s}{15} = \frac{i}{0.9-x} \cdot \frac{f_s}{15 f_c} \cdot d^2 \cdot f_c = m' \cdot d^2 \cdot f_c.$$

The coefficients m and m', i.e., the resisting moments per unit fiber stress for a slab of thickness $d=1$, depend only upon the percentages of reinforcement a' and a''. If these quantities are laid off on a pair of coordinate axes, the quantities m and m' will be represented by a series of curves; Melan has thus constructed a graphic chart for giving the above coefficients directly (Plate I).* The curves for m' are nearly straight, parallel lines and their intersections with the corresponding m-curves lie on a straight line whose position and direction depend upon the ratio of $f_s : f_c$. The chart is drawn for $f_s : f_c = 30$.

The chart proves particularly useful in designing the reinforcement for sections subjected to reversal of moments. Let these extreme moments be $M_1 = m_1 d^2 f_c$ and $M_2 = -m_2 d^2 f_c$.

If the curve m_2 is drawn in the chart with the coordinate axes interchanged, its intersection with m_1 determines the percentages of reinforcement (see Fig. 41). Three different cases may thus occur. The two curves may intersect each other in the portions m', or in the portions m, or the m-branch of one curve may intersect the m'-branch of the other. In the first case, if the percentage of the reinforcement is fixed according to the intersection-point of the two curves, the steel will be stressed to the limit while the concrete remains understressed. In the second case, the strength of the concrete is fully utilized, but that of the steel is not exhausted. In the third case, if, for instance, m_1 intersects m_2', for one extreme moment M_1 the concrete is stressed to the limit, for the other extreme moment M_2 the steel is stressed to the limit.

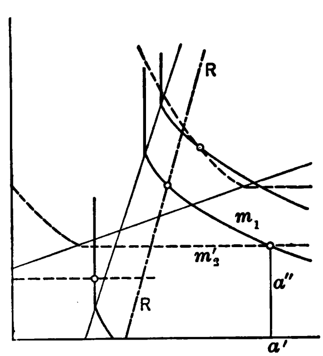

FIG. 41.—Method of Using Reinforcement Chart.

* *Melan:* "Hilfstafel zur Berechnung doppelt armierter Balken, Platten und Gewölbe."—Techn. Blätter, Prague, 1907. Also "Handbuch für Eisen-Betonbau," vol. 1, p. 266.

In cases 2 and 3, however, the percentages of reinforcement given by the points of intersection do not constitute the minimum requirements of steel (with the depth d fixed), but the quantity of metal necessary may be reduced by moving along the m-curve corresponding to the larger moment toward the line marked RR; this being the locus of points at which the curves have tangents at a slope of 45°, i.e., the points at which $(a'+a'')=$ a minimum. In all such cases, the most economic design is afforded by an upper and a lower reinforcement of approximately equal amounts.

The curves of the above chart may be transferred to tracing cloth, and this copy may be laid reversed upon the chart in order to obtain a solution for all possible cases. (In Plate 1 the reversed curves have been inserted in red to facilitate the use of the chart. This chart is useful for designing reinforced concrete beams and slabs, as well as arch-ribs.)

Arches in which the various sections are very unequally stressed, will, in an economical design, receive a varying reinforcement. The area of steel is varied by adding or omitting some of the reinforcing bars; but care should be taken that the bars are extended beyond the sections at which they are required for a sufficient length to develop the necessary adhesion. This length is determined, for circular bars of diameter d, by the familiar relation

$$l=\frac{1}{4}\frac{f_s}{u}\cdot d,$$

where f_s is the intensity of stress in the steel and u is the allowable shear intensity between the steel and the concrete.

EXAMPLE.—In an arch rib of 1 ft. depth, the extreme moments (per 1 ft. width) at the most severely stressed section are $M_1=14,880$ ft.-lbs. and $M_2=-9720$ ft.-lbs. The axial thrusts corresponding to the two cases of loading amount to 9500 lbs. and 8100 lbs. respectively, so that the stresses from this source, taking the steel into account, scarcely amount to 50 lbs. per sq.in. We deduct this from the allowable compression in concrete (assumed at 500 lbs. per sq.in.) and therefore use $f_c=450$ in

the calculations. Let the allowable stress in the steel be $f_s = 13{,}500 = 30f_c$. We then have

$$m_1 = \frac{14880}{d^2 \cdot f_c} = \frac{14880}{144 \times 450} = 0.23,$$

and

$$m_2 = \frac{9720}{144 \times 450} = 0.15.$$

For the point of intersection of these two curves we find from the chart (Plate I)

$$a' = 2.11 \quad \text{and} \quad a'' = 0.623.$$

These percentages of reinforcement correspond to a ratio of $f_s : f_c = 14.36$ by curve m_1, and to a ratio of $f_s : f_c = 45.78$ by curve m_2'. The stresses are therefore: (lbs. per sq.in).

$$\begin{cases} \text{in the concrete, uppermost fiber, } 450, \\ \text{in the concrete, lowest fiber, } \dfrac{13500}{45.78} = 295, \\ \text{in the upper reinforcement, } 13500, \\ \text{in the lower reinforcement, } 14.36 \times 450 = 6460. \end{cases}$$

If, however, the percentages are chosen on the curve $m_1 = 0.23$ nearer to the optimum line RR, so that both of the reinforcements are equal in amount, then $a' = a'' = 1.16$ per cent and the stresses produced by M_1 are

$$\begin{cases} \text{in the concrete, uppermost fiber, } 450, \\ \text{in the lower reinforcement, } 25.07 \times 450 = 11{,}290. \end{cases}$$

The stresses produced by M_2 will bear to these the ratio $\dfrac{M_2}{M_1} = \dfrac{0.15}{0.23} = 0.65$, and will therefore be

$$\begin{cases} \text{in the concrete, lowest fiber, } 0.65 \times 450 = 292, \\ \text{in the upper reinforcement, } 0.65 \times 11{,}290 = 7340. \end{cases}$$

Whereas in the first case the total steel reinforcement was $2.11 + 0.62 = 2.73$ per cent, in the second case it is only $2 \times 1.16 = 2.32$ per cent. The symmetrical reinforcing thus results in a saving of material and the resistances of the steel in the upper and lower layers are more equally utilized.

At another section of the same arch, the effective moments wer $M_1=10368$ ft.-lbs. and $M_2=-8400$ ft.-lbs.; hence, $m_1=\frac{10368}{144\times450}=0.16$ and $m_2=\frac{8400}{144\times450}=0.13$. These two curves intersect at a point whose coordinates are $a'=0.615$ and $a''=0.537$ per cent. A reduction in the amount of steel is here impossible since the steel in both reinforcements is already stressed to 13,500 lbs. per sq.in.

Choosing, for the reinforcement, round rods of $\frac{1}{2}$ in. diameter, there will be required, in the first section considered, 8 rods (per foot) on each side of the axis; at the other section, there are needed 5 rods below and 4 above the neutral axis.

EXAMPLES

I. DESIGN OF AN ARCH BY THE ANALYTIC METHOD. REINFORCED CONCRETE ARCH—MELAN TYPE (POLCEVERA BRIDGE AT GENOA.)

The five arch spans of this bridge, separated by strong, deep piers, have a clear span of 69 ft. and a rise of 7.5 ft. The radial depth is 1.48 ft. at the crown and 2.6 ft. at the ends. The reinforcement consists of braced arch ribs spaced 3.3 ft. apart, whose chords are formed of two angles $2\frac{3}{4} \times 3\frac{1}{2} \times \frac{5}{16}$ ins. in cross-section.

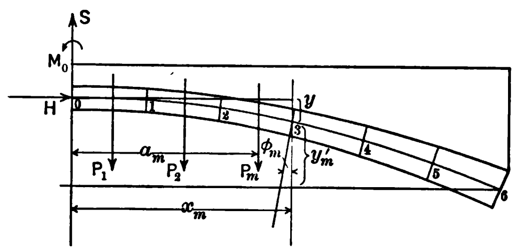

FIG. 42.—Design of an Arch by the Analytic Method.

For the purpose of the design, the half of the arch is divided into six segments of equal length, $\Delta s = 6.0$ ft. The points of division on the arch-axis are numbered from 0 at the crown to 6 at the end. The coordinates of these points, x and y, measured from the crown, also the depth d and the sectional quantities A and I, are contained in the following table. (See Fig. 42.)

Dimensional Quantities (per ft. width)

(All units reduced to ft. and lbs.)

Depth of rib $= d$. Depth of steel ribs $= d - 0.26$

Section of concrete $= A_c = 1 \cdot d$ $\qquad I_c = \frac{1}{12} \cdot 1 \cdot d^3$

Section of steel $= A_s = 0.016$ $\qquad I_s$

$A = A_c + 15 \cdot A_s$ $\qquad I = I_c + 15 \cdot I_s$

Point.	0	1	2	3	4	5	6
x	0	6.00	12.00	17.98	23.85	29.56	35.00
y	0	0.16	0.68	1.57	2.92	4.79	7.28
d	1.48	1.48	1.48	1.48	1.60	1.97	2.59
$\sin\phi$	0	.056	.117	.187	.268	.364	.447
$\cos\phi$	1	.998	.993	.982	.963	.931	.894
A_c	1.48	1.48	1.48	1.48	1.60	1.97	2.59
$15A_s$	.24	.24	.24	.24	.24	.24	.24
A	1.72	1.72	1.72	1.72	1.84	2.21	2.83
I_c	.270	.270	.270	.270	.342	.634	1.442
$15I_s$	.072	.072	.072	.072	.091	.149	.293
I	.342	.342	.342	.342	.433	.783	1.735

CALCULATION OF THE COEFFICIENTS $\alpha, \beta, \phi, \epsilon$, OF EQS. (19).

Point.	0	1	2	3	4	5	6
$1/I$	2.92	2.92	2.92	2.92	2.31	1.28	0.58
y/I	0	0.48	2.00	4.60	6.74	6.10	4.20
y^2/I	0	0.08	1.35	7.22	19.66	29.20	30.60
x^2/I	0	106	422	946	1310	1109	703
$\dfrac{\sin^2\phi}{A}$	0	.002	.008	.020	.039	.060	.071
$\dfrac{\cos^2\phi}{A}$	.580	.579	.574	.562	.504	.393	.283
$\dfrac{\sin\phi\cos\phi}{A}$	0	.032	.068	.107	.140	.153	.141

By the formulæ (19) we then obtain,

$$\beta = 227.8 \text{ ft.}^{-2};$$
$$\alpha = -66.4 \text{ ft.}^{-3};$$
$$\phi = 42.44 \text{ ft.}^{-4};$$
$$\epsilon = 12812 \text{ ft.}^{-2}.$$

1. EFFECT OF DEAD LOAD

(*a*) THE ARCH PROPER. (DENSITY OF THE REINFORCED CONCRETE = 150)

Point.	0	1	2	3	4	5	6
d	1.48	1.48	1.48	1.48	1.60	1.97	2.59

Δs	6.00	6.00	6.00	6.00	6.00	6.00
Weight of rib, P'	1330	1330	1330	1388	1610	2010
a	3.0	9.0	15.0	20.9	26.8	32.4

	0	1	2	3	4	5	6
$\mathfrak{M}'$	0	3990	15960	35810	63240	98500	141800
$\dfrac{1}{I}\mathfrak{M}'$	0	11660	46700	104750	146000	125800	81700
$\dfrac{y}{I}\mathfrak{M}'$	0	1900	31920	165000	427000	601000	595000
$\dfrac{\sin\phi\cdot\cos\phi}{A}(P_1+\ldots P_m)$	0	43	180	426	753	1075	1276

Hence, by Eqs. (20),

$$\begin{cases} A_1 = -4{,}584{,}400 + 9{,}300 = -4{,}575{,}100 \dfrac{\text{lbs.}}{\text{ft.}^2} \\ A_2 = 1{,}435{,}900 \text{ lbs./ft.}^3 \end{cases}$$

The equations of condition (18) then become

$$\begin{cases} -4{,}575{,}100 + 227.8H - 66.4M_0 = 0, \\ 1{,}435{,}900 - 66.4H + 42.44M_0 = 0. \end{cases}$$

Hence we obtain

$$H' = 18{,}740 \text{ lbs.},$$

$$M_0' = -4{,}460 \text{ ft.-lbs.}$$

(*b*) DEAD LOAD OF THE SPANDREL FILLING. (DENSITY = 112)

Point.	0	1	2	3	4	5	6
Depth of Filling...........	0.98	1.15	1.64	2.53	3.84	5.61	7.87

Δx..........................	6.04	6.07	6.00	5.94	5.87	5.68
Weight, P''.................	724	951	1405	2140	3120	4300
a..........................	3.09	9.25	15.32	21.29	27.17	32.82

Point.	0	1	2	3	4	5	6
$\mathfrak{M}''$......................	0	2103	9,065	22,760	46,340	83,500	137,800
$1/I \cdot \mathfrak{M}''$..................	0	6140	26,500	66,600	106,800	106,500	79,400
$y/I \cdot \mathfrak{M}''$..................	0	1010	18,130	104,800	312,000	509,200	57,900
$\dfrac{\sin\phi \cdot \cos\phi}{A}(P_1 + \ldots P_m)$	0	24	110	330	730	1,280	1,780

Hence, by Eqs. (20),

$$\begin{cases} A_1 = -3{,}698{,}000 + 10{,}000 = -3{,}688{,}000 \text{ lbs./ft.}^2 \\ A_2 = 1{,}063{,}000 \text{ lbs./ft.}^3 \end{cases}$$

The equations of condition (18) then become

$$\begin{cases} -3{,}688{,}000+227.8H-66.4M_0 =0, \\ 1{,}063{,}000-\ 66.4H+42.44M_0=0. \end{cases}$$

Hence

$$\begin{cases} H''=16{,}300 \text{ lbs.}, \\ M_0''=480 \text{ ft.-lbs.} \end{cases}$$

It is assumed that, by a partial suspension of the falsework from the steel ribs, at least one-third of the weight of the arch proper is carried directly by the reinforcement. Hence, with the total dead-weight acting, the concrete-steel arch will be subjected to the following:

$$\begin{cases} \text{at the crown}: \tfrac{2}{3}H'+H'' \text{ and } \tfrac{2}{3}M_0'+M_0'', \\ \text{at any point } m: N=(\tfrac{2}{3}H'+H'')\cos\phi_m+[\tfrac{2}{3}(P_1+\ \ldots\ P_m') \\ \qquad\qquad +(P_1''+\ldots\ P_m'')]\cdot\sin\phi_m \\ \qquad M=(\tfrac{2}{3}M_b'+M_b'')-(\tfrac{2}{3}H'+H'')y_m+(\tfrac{2}{3}M_0'+M_0''). \end{cases}$$

These values are calculated in the following table:

Point.	0	1	2	3	4	5	6
$(\frac{2}{3}H'+H'')\cos\phi$...	28,790	28,700	28,600	28,270	27,700	26,800	25,740
$(\frac{2}{3}\Sigma P'+\Sigma P'')\sin\phi$.	0	90	400	1,070	2,350	4,720	8,350
N............(lbs.)	28,780	28,790	29,000	29,340	30,050	31,520	34,090
M.........(ft.-lbs.)	−2,500	−2,520	−2,250	−1,160	+1,875	+8,420	+20,150

DEAD LOAD FIBER STRESSES

(a) Concrete

$$f_c=\frac{N}{A}\mp\frac{M}{I}\cdot\frac{d}{2}.$$

Point.	0	1	2	3	4	5	6
$\frac{N}{A}$................(lbs./in.²)	116	117	117	119	113	99	84
$\frac{M}{I}\cdot\frac{d}{2}$.....................	− 38	− 37	− 34	− 17	24	73	104
f_c upper fiber............	154	154	151	136	89	26	− 20
f_c lower fiber............	78	80	83	102	137	172	188

(*b*) Steel

The steel rib first takes up one-third of the weight of the arch. The stresses thus produced may be calculated with sufficient accuracy by

$$f_s'=\frac{1}{3}\left[\frac{N'}{A_s}\mp\frac{M'}{I_s}\left(\frac{d}{2}-0.13\right)\right].$$

Here,

$$\begin{cases} N'=H'\cos\phi_m+(P_1+\ .\ .\ .\ P_m')\sin\phi_m, \\ M'=M_b'-H'.y+M_0'. \end{cases}$$

To this initial stress f_s' must be added the stress

$$f_s''=15\left[\frac{N}{A}\mp\frac{M}{I}\left(\frac{d}{2}-0.13\right)\right],$$

wherein N and M must be substituted from a preceding table.

Point.	0	1	2	3	4	5	6
$H'\cdot\cos\phi$	18,740	18,700	18,590	18,400	18,050	17,450	16,750
$(P_1'+\ .\ .\ .\ P_m')\sin\phi$	0	70	310	740	1,440	2,540	4,040
N' (lbs.)	18,740	18,770	18,900	19,140	19,490	19,990	20,790
M' (ft.-lbs.)	−4,470	−3,550	−1,160	+1,850	+4,070	+4,050	+900
$\frac{N'}{A_s}$ (lbs./in.²)	8,150	8,160	8,210	8,310	8,490	8,700	9,330
$\frac{M'}{I_s}\left(\frac{d}{2}-.13\right)$	−3,880	−3,070	−1,010	+1,610	+3,160	+2,380	+370
$f_s'\left(\frac{\text{lbs.}}{\text{in.}^2}\right)$ lower fiber	4,010	3,740	3,070	2,230	1,780	2,110	2,990
upper fiber	1,422	1,690	2,400	3,310	3,880	3,690	3,230
N/A	116	117	118	119	114	99	84
$\frac{M}{I}\left(\frac{d}{2}-.13\right)$	−31	−30	−27	−14	20	63	93
$f_s''\left(\frac{\text{lbs.}}{\text{in.}^2}\right)$ upper fiber	2,205	2,205	2,175	1,995	1,410	540	−135
lower fiber	1,275	1,305	1,365	1,575	2,010	2,430	2,655
$f_s=f_s'+f_s''$ upper fiber	6,215	5,945	5,245	4,225	3,190	2,650	2,855
lower fiber	2,697	2,995	3,765	4,885	5,890	6,120	5,885

II. FULL SPAN LOAD OF 305 LBS./FT.²

(Prescribed Loading)

Point.	0	1	2	3	4	5	6
x	0	6.00	12.00	17.98	23.85	29.56	35.00
$\mathfrak{M}=\frac{1}{2}\cdot305\cdot x^2$	0	5,500	22,100	49,800	87,500	134,000	188,000
$\mathfrak{M}/I$	0	16,050	65,000	145,800	202,400	170,550	108,000
$y/I\cdot\mathfrak{M}$	0	2,670	44,200	229,000	592,000	818,000	786,000
$\frac{\sin\phi\cdot\cos\phi}{A}(P_1+\ .\ .\ .\ P_m)$	0	60	250	590	1,035	1,410	1,545

Hence
$$\begin{cases} A_1 = -6{,}257{,}100 + 12{,}400 = -6{,}234{,}700, \\ A_3 = 1{,}970{,}000. \end{cases}$$

The equations of condition (18) are then

$$\begin{cases} -6{,}234{,}700 + 227.8H - 66.4M_0 = 0, \\ 1{,}972{,}400 - 66.4H + 42.44M_0 = 0. \end{cases}$$

These yield

$$\begin{cases} H = 26{,}500 \text{ lbs.}, \\ M_0 = -3000 \text{ ft.-lbs.} \end{cases}$$

Point.	0	1	2	3	4	5	6
$H \cdot \cos\phi$	26,500	26,450	26,300	26,050	25,550	24,700	23,700
$(P_1 + \ldots P_m) \sin\phi$..	0	100	430	1,030	1,960	3,300	4,810
N............. (lbs.)	26,500	26,550	26,730	27,080	27,510	28,000	28,510
$M = M_b - Hy + M_0$...	−3,000	−1,800	+1,210	+4,950	+6,980	+4,100	−7,680
N/A...............	107	108	108	110	103	88	70
$M/I \cdot d/2$...........	− 45	− 27	+ 19	74	90	36	− 40

STRESSES IN THE CONCRETE

Upper fiber.........	152	135	89	36	13	52	110
Lower fiber.........	62	81	127	184	193	124	30

STRESSES IN THE STEEL

Upper fiber.........	2,160	1,950	1,380	725	425	850	1,580
Lower fiber.........	1,050	1,280	1,860	2,560	2,660	1,770	510

III. HALF-SPAN LOADED WITH 305 LBS./FT.[2]

H and M_0 receive one-half the values produced by a full-span load, i.e.,

$$H=\tfrac{1}{2}\cdot(26,500)=13,250 \text{ lbs.},$$

$$M_0=-\tfrac{1}{2}(3000)=-1500 \text{ ft.-lbs.}$$

To determine S, we use the third of the Eqs. (18). The coefficient A_2 is calculated from the following values:

Point.	0	1	2	3	4	5	6
$\mathfrak{M}\frac{x}{I}$	0	57,600	780,000	2,620,000	4,820,000	5,040,000	3,780,000

Hence $\quad A_2=-\tfrac{1}{2}(45,850,000)=-22,925,000$

$$-22,925,000+12,812S=0, \quad \therefore \quad S=1790 \text{ lbs.}$$

For the loaded side:

$$M=M_b-H\cdot y-S\cdot x+M_0, \quad N=H\cdot\cos\phi+(P_1+P_2+\ldots P_m-S)\sin\phi.$$

Point.	0	1	2	3	4	5	6
M_b..........	0	5,540	22,150	49,470	87,440	134,340	188,300
$H\cdot y$........	0	2,180	8,950	20,900	38,750	63,600	96,500
$S\cdot x$.........	0	10,740	21,460	32,150	42,600	52,800	62,510
M_0.........	− 1,500	− 1,500	− 1,500	− 1,500	− 1,500	− 1,500	− 1,500
M..........	− 1,500	− 8,880	− 9,760	− 4,810	− 4,590	+16,440	+27,790
$H\cdot\cos\phi$.....	13,250	13,225	13,150	13,025	12,775	12,350	11,850
$(\Sigma P-S)\sin\phi$	0	3	225	698	1,485	2,650	4,015
N..........	13,250	13,228	13,375	13,723	14,260	15,000	15,865

For the unloaded side:

$$M=-H\cdot y+S\cdot x+M_0, \quad N=H\cdot\cos\phi+S\cdot\sin\phi.$$

Point.	0	1	2	3	4	5	6
M..........	− 1,500	+ 7,060	+11,010	+ 9,750	+ 2,350	−12,300	−35,490
N..........	13,250	13,330	13,360	13,350	13,250	12,990	12,650

STRESSES IN THE CONCRETE

Point.	Unloaded Side.						Loaded Side.						
	6′	5′	4′	3′	2′	1′	0	1	2	3	4	5	6
N/A....	31	41	50	54	54	54	54	53	54	55	54	47	39
$\frac{M}{I}\cdot\frac{d}{2}$....	−183	−107	31	147	165	106	−22	133	−146	−72	+59	+141	+143
f_c upper	214	148	19	−93	−111	−52	76	186	200	127	−5	−94	−104
f_c lower	−152	−66	81	201	219	160	32	−80	−92	−17	113	188	182

(f_c in lbs. per sq.in.; − sign denotes tension)

STRESSES IN THE STEEL

Point.	Unloaded Side.						Loaded Side.						
	6′	5′	4′	3′	2′	1′	0	1	2	3	4	5	6
f_s upper	2930	1990	330	−1000	−1220	−500	1080	2450	2660	1720	70	−1140	−1500
f_s lower	−1990	−780	1140	2620	2850	2120	530	−840	−1040	−60	1550	2560	2500

IV. SPECIAL LOADING WITH ROAD-ROLLER

The special loading consists of an 11-ton axle load (=22,000 lbs.) placed at the crown, and a uniform load of 122 lbs. per sq.ft. over the remainder of the span. It is assumed that the weight of the axle load is distributed over a width of 4 ft. and a length of 2.25 ft. Hence the intensity of the loading at the crown is $\frac{22000}{4\times 2.25}=2444$ lbs./ft.². Deducting the full-span load of 122 lbs./ft.², there remains at the crown a special load of 2322 lbs./ft.² for a length of 2.25 ft.

The resultant stresses are determined by reducing the values obtained for the full-span load of 305 lbs./ft.² (under II) in the ratio $\frac{122}{305}=0.4$, and then adding in the effect of the crown load of $2.25\times 2322=5225$ lbs. The latter stresses are calculated as follows:

Load for one-half the span, at the crown,

$$P=\tfrac{1}{2}\cdot 5225=2612 \text{ lbs.};\quad a=0.56 \text{ ft.}$$

Point.	0	1	2	3	4	5	6
$\mathfrak{M}$	0	14,440	30,400	46,300	61,900	77,100	91,500
$\mathfrak{M}/I$	0	42,300	89,000	135,500	143,100	97,900	52,400
$\mathfrak{M}\cdot y/I$	0	6,940	60,200	214,000	418,200	470,000	382,000

Hence
$$\begin{cases} A_1=-4,102,600+4600=-4,098,000, \\ A_3=1,620,000 \text{ lbs./ft.}^3 \end{cases}$$

Consequently, by Eqs. (18),

$$\begin{cases} -4,098,000+227.8H-66.4M_0=0, \\ 1,620,000-66.4H+42.44M_0=0. \end{cases}$$

Affixing the subscript (p) for the effect of the concentration P, we then obtain

$$H_p=12,600 \text{ lbs.};\quad M_{0,p}=-18,440 \text{ ft.-lbs.}$$

Adding to this the effect of the full-span load,

$$0.4H_{(305)}=10,600 \text{ lbs.},\quad 0.4\ M_{0,(305)}=-1200,$$

we obtain the resultant values

$$H = 23{,}200 \text{ lbs.} \quad \text{and} \quad M_0 = -19{,}640 \text{ ft.-lbs.}$$

Point.	0	1	2	3	4	5	6
$M_b - H_p \cdot y + M_{0,p}$....	−18,440	− 6,020	+ 3,500	+ 8,050	+ 2,480	− 1,790	−18,600
$0.4M_{(305)}$..........	− 1,200	− 720	+ 480	+ 1,980	+ 2,790	+ 1,640	− 3,070
M..........(ft.-lbs.)	−19,640	− 6,740	+ 3,980	+10,030	+ 5,270	− 150	−21,670
$H_p \cdot \cos\phi + P \cdot \sin\phi$..	12,600	12,740	12,830	12,860	12,850	12,700	12,460
$0.4N_{(305)}$..........	10,600	10,620	10,690	10,830	11,000	11,200	11,400
N.............(lbs.)	23,200	23,360	23,520	23,690	23,850	23,900	23,860

STRESSES IN THE CONCRETE

Point.	0	1	2	3	4	5	6
N/A..............	94	94	95	96	90	75	58
$\frac{M}{I}\cdot\frac{d}{2}$..............	− 294	− 101	+ 60	+ 151	+ 68	− 1	− 112
f_c upper fiber.....	388	195	35	− 55	22	76	170
f_c lower fiber.....	− 200	7	155	247	158	74	− 54

STRESSES IN THE STEEL

Point.	0	1	2	3	4	5	6
f_s upper fiber.....	5040	2660	680	− 430	500	1140	1520
f_s lower fiber.....	−2220	170	2180	3290	2200	1110	− 630

V. EFFECTS OF TEMPERATURE

The horizontal thrust produced by a temperature change of $T°$ is

$$H_t = \frac{3}{2} \cdot \frac{E\omega T l}{\Delta s} \cdot \frac{\phi}{\beta\phi - \alpha^2}.$$

The crown-moment

$$M_{0,t} = -\frac{\alpha}{\phi} \cdot H_t.$$

Here ω denotes the coefficient of linear expansion, Δs the length of each division of the arch, β, α, ϕ, the arch-constants defined by Eqs. (19) and calculated above, and E the coefficient of elasticity of the concrete.

Substituting $E = 2{,}000{,}000$ lbs. per sq.in., $\omega = 0.0000068$, $T = \pm 26°$F, $E\omega T = 51{,}000$ lbs./ft.², we obtain

$$\begin{cases} H_t = \dfrac{3}{2} \cdot \dfrac{51000 \times 70}{6} \cdot \dfrac{42.44}{227.8 \times 42.44 - 66.4^2} = 7200 \text{ lbs.} \\[2ex] M_{0,t} = -\dfrac{66.4}{42.44} H_t = -1.56\, H_t. \end{cases}$$

We then obtain the moments and axial thrusts at the individual sections by the relations:

$$\begin{cases} M = -H_t \cdot y + M_0, \\ N = H_t \cdot \cos \phi. \end{cases}$$

Point.	0	1	2	3	4	5	6
M_t.... (ft.-lbs.)	+11,230	+10,050	+6,370	− 95	−9,800	−23,300	−41,200
N_t...... (lbs.)	7,200	7,180	7,150	7,060	6,740	6,700	6,420
N/A.. $\left(\frac{\text{lbs.}}{\text{in.}^2}\right)$	29	29	29	29	26	21	16
$\frac{M}{I}\cdot\frac{d}{2}$.. $\left(\frac{\text{lbs.}}{\text{in.}^2}\right)$	169	151	96	− 1	− 126	− 202	− 212

STRESSES IN THE CONCRETE FOR $T = \pm 26°$ F.

	0	1	2	3	4	5	6
f_c upper...	∓ 140	∓ 122	∓ 67	± 30	± 152	± 223	± 228
f_c lower...	± 198	± 180	± 125	± 28	∓ 100	∓ 181	∓ 196

STRESSES IN THE STEEL FOR $T = \pm 26°$ F.

	0	1	2	3	4	5	6
f_s upper...	∓2,250	∓1,420	∓ 750	± 480	±1,980	±2,950	± 3,100
f_s lower...	±2,520	±2,300	±1,610	± 410	∓1,180	∓2,330	∓ 2,620

(Fiber stresses in lbs. per sq.in.; −denotes tension.)

Combining the effects of dead load, live load and temperature, we obtain:

TOTAL STRESSES IN THE CONCRETE

Point.		0	1	2	3	4	5	6
Dead load	upper fiber	154	154	151	136	89	26	− 20
	lower fiber	78	80	83	102	137	172	188
Full load of 305 lbs./ft.²	upper fiber	306	287	240	172	102	78	90
	lower fiber	140	161	210	286	330	300	218
Right half loaded with 305 lbs./ft.²........	upper fiber	230	340	351	263	84	− 68	−124
	lower fiber	110	0	− 9	85	250	360	370
Left half loaded with 305 lbs./ft.²........	upper fiber	230	102	40	43	108	164	194
	lower fiber	110	240	302	303	218	106	36
Road-roller (11-ton)+ 122 lbs./ft.² total ...	upper fiber	542	349	186	81	111	102	150
	lower fiber	−122	87	238	349	295	246	134
Temperature effect ...	upper fiber	∓140	∓122	∓ 67	± 30	±152	±223	±228
	lower fiber	±198	±180	±125	± 28	∓100	∓181	∓196

The maximum stresses occur:

1. At the crown, under the action of the 11-ton axle load with 122 lbs. per sq.ft. over the whole span and maximum fall of temperature.

$$\begin{cases} f_{c,\,\max}=542+140=682 \text{ lbs. per sq.in (compression in upper fiber)}, \\ f_{c,\,\min}=-122-198=-320 \text{ lbs. per sq.in. (tension in lower fiber)}. \end{cases}$$

This large tension in the concrete shows the above method of calculation to be incorrect for this section. We must therefore compute the stresses on the second assumption (Phase II) by which the tensile resistance of the concrete is discounted. The resulting forces acting on the crown section are:

	M, ft.-lbs.	N, lbs.
From the dead load....................	− 2,500	28,780
From the live load.....................	−19,640	23,200
From temperature....................	−11,230	− 7,200
	$M=-33,370$	$N=44,780$

Hence $$p_n=\frac{M}{N}=\frac{33370}{44780}=0.745 \text{ ft.}$$

Referring back to Chapter II, we write $E_{ct}=0.4\ E_{cp}$, hence $m=0.4$. Substituting this value, also $d=1.48$ and $j=0.5\ d$, in Eq. (7), we obtain the following equation for determining the distance x of the neutral axis from the compression fiber:

$$x^3+0.016x^2+6.17x-6.36=0.$$

Hence $$x=0.909 \text{ ft.}$$

Eq. (7a) then gives

$$44780=[\tfrac{1}{2}(\overline{0.909}^2-0.4\times\overline{0.57}^2)1728+1728\times0.24\times0.17]f_u$$

$$\therefore\ f_u=66.6 \text{ lbs. per sq.in. (at 1 in. from neutral axis)},$$

and the maximum compression in the concrete becomes

$$f_c=0.909\times12\times66.6=726 \text{ lbs./sq. in.};$$

the maximum tension in the concrete:

$$f_a = 0.4 \times 0.57 \times 12 \times 66.6 = 181 \text{ lbs./sq.in.}$$

The maximum stresses in the steel become:

$$\begin{cases} \text{Compression, } f_{sp} = 15 \times 9''.34 \times 66.6 + 4010 = 13350. \\ \text{Tension, } \quad f_{st} = -15 \times 5''.24 \times 66.6 + 1422 = -3820. \end{cases}$$

If we completely neglect the tensile resistance of the concrete, hence substituting $m=0$ in Eq. (7), the stress in the concrete increases to 824 lbs. per sq.in.; the compression in the steel to 14,200; the tension to 8440 lbs. per sq.in.

Actually, however, the above stresses will hardly occur, since the effect of the axle load has been too severely figured. The 11-ton concentration has been considered as covering a width of only 4 ft.; but, as the width of the roadway, 49 ft., cannot accommodate more than seven wagons or road-rollers, the 11-ton load will actually cover a width of $\frac{49}{7} = 7$ ft.

2. At the ends, the maximum stresses occur with one-half of the span covered with 305 lbs./sq.ft. and the greatest fall in temperature. We first obtain, on the basis of Phase I,

$$\begin{cases} f_{max} = 370 + 196 = 566 \text{ lbs./in.}^2 \text{ (compression in lower fiber),} \\ f_{min} = -124 - 228 = -352 \text{ lbs./in.}^2 \text{ (tension in upper fiber).} \end{cases}$$

In this case, as before, we should discount the tensile resistance of the concrete in order to obtain the stresses more correctly. We proceed as follows:

We have

	M, ft.-lbs.	*N*, ft.-lbs.
From the dead load	+20,150	34,090
From the live load	+27,790	15,865
From the temperature	+41,200	− 6,420
	$M = +89,140$	$N = 43,535$

hence $$p_n = \frac{89140}{43535} = 2.05 \text{ ft.}$$

The section of the reinforcement at the ends, strengthened by the addition of a $7\frac{1}{2}\times\frac{3}{8}$-in. cover plate, gives $15A_s=0.426$ sq.ft., $15I_s=0.547$; with these values and $m=0.4$, Eq. (7) becomes

$$x^3+2.25x^2+29.92x-50.18=0$$

$$\therefore\quad x=1.424 \text{ ft.}$$

Consequently, by Eq. (7*a*)

$$43{,}535=[\tfrac{1}{2}(\overline{1.424}^2-0.4\times\overline{1.166}^2)1728+1728\times0.426\times0.129]\cdot f_u,$$

$$\therefore\quad f_u=31.6 \text{ lbs./in.}^2 \text{ at 1 in. from axis.}$$

The maximum compression in the concrete then becomes

$$f_c=1.424\times12\times31.6=540 \text{ lbs. per sq.in.}$$

The maximum tension in the concrete

$$f_{ct}=0.4\times1.166\times12\times31.6=178 \text{ lbs. per sq.in.}$$

The maximum stresses in the steel are:

$$\begin{cases}\text{compression, } f_{sp}=15\times12\times1.325\times31.6+3230=10{,}770,\\ \text{tension ,}\quad f_{st}=-15\times12\times1.07\times31.6+2990=-3105.\end{cases}$$

At all other sections, there appear very slight tensile stresses or none at all; and the maximum compression in the concrete remains below 570 lbs. per sq.in.

TOTAL STRESSES IN THE STEEL

(Lbs. per sq.in.)

Point.		0	1	2	3	4	5	6
Dead load.......	u. fiber	6215	5945	5245	4225	3190	2650	2855
	l. fiber	2697	2995	3765	4885	5890	6120	5885
Live load (305	u. fiber	8375	7895	6625	4950	3615	3500	4435
lbs./ft.²)........	l. fiber	3647	4275	5625	7445	8550	7890	6395
Live load on right	u. fiber	7295	8395	7905	5945	3260	1510	1355
half...........	l. fiber	3227	2155	2725	4825	7440	8680	8385
Live load on left	u. fiber	7295	5445	4025	3225	3520	4640	5785
half...........	l. fiber	3227	5115	6615	7505	7030	5340	3895
Special load (11 tons	u. fiber	11255	8605	5925	3795	3690	3790	4375
+122 lbs./ft.²)..	l. fiber	477	3165	5945	8175	8090	7230	5255
Temperature....	u. fiber	∓2250	∓1420	∓ 750	± 480	±1980	±2950	±3100
	l. fiber	±2520	±2300	±1610	± 410	±1180	±2330	±2620

2. DESIGN OF AN ARCH BY THE GRAPHIC METHOD

Hingeless Arch

(Chauderon-Montbenon Bridge at Lausanne)

By the graphic method developed above (Chapter IV*b*), we determine the *influence-lines* for the moments at the different arch-sections.

In the hingeless arch, the moment at any point C (Fig. 15) is given by

$$M = M_b - H \cdot y - X_1 \cdot x - X_2.$$

Here M_b denotes the moment in a simple beam under the same loading, H the horizontal thrust, X_1 the difference between the vertical end-reactions in the arch and the corresponding reactions in a simple beam, and X_2 a moment defined by $X_2 = H \cdot z_0$. The axis of abscissæ must be so located as to render the sum of the static moments about that axis of the arch-elements, loaded with the reciprocals $\frac{1}{I}$, equal to zero; hence $\Sigma \frac{y}{I} \cdot \Delta s = 0$. The influence-lines for H, X_1 and X_2 may be constructed as the funicular polygons for the arch-elements loaded successively with the "weights" $w = \frac{y}{I} \cdot \Delta s$, $w' = \frac{x}{I} \cdot \Delta s$ and $w'' = \frac{1}{I} \cdot \Delta s$. In Fig. 43, diagrams (4), (7) and (8) show these funicular polygons in dash-and-dot lines. First, by aid of the force-polygon for the weights w'' (1) and the funicular polygon for the same weights applied horizontally (2), the position of the axis of abscissæ is determined. Then the weights w and w' are computed, and, by aid of their force-polygons, (3) and (6), the funicular polygons (4) and (7) are constructed. The pole-distance in (3) is chosen arbitrarily; and the scale-unit for H, that is the implied magnitude of the load G, is obtained by adding a small correction to the intercept $2\overline{m_0 m}$ of the funicular polygon (5) which is constructed for the weights w applied horizontally. The pole-distance in (6) is also arbitrary; the ordinates of the X_1-polygon must be measured by a scale-unit equal to the length $2\overline{n_0 n}$ intercepted on the axis of the ordinates between the first and last sides of the polygon. On the other hand, in the force-polygon (1) for w'', the pole-distance is taken $= \frac{1}{2}\Sigma w''$, so that the outside ray is at 45°; the ordinates of the funicular polygon (8) then give the quantity X_2 to twice the scale of lengths. In order, next, to obtain the moments $H \cdot y$ and $X_1 \cdot x$ to the same scale, the respective funicular polygons H and X_1 should be reduced in the ratio $\frac{y}{2}$ and $\frac{x}{2}$, which may be accomplished by changing the respective pole-distances in the ratio $y : \overline{mm_0}$ and $x : \overline{nn_0}$. Accordingly,

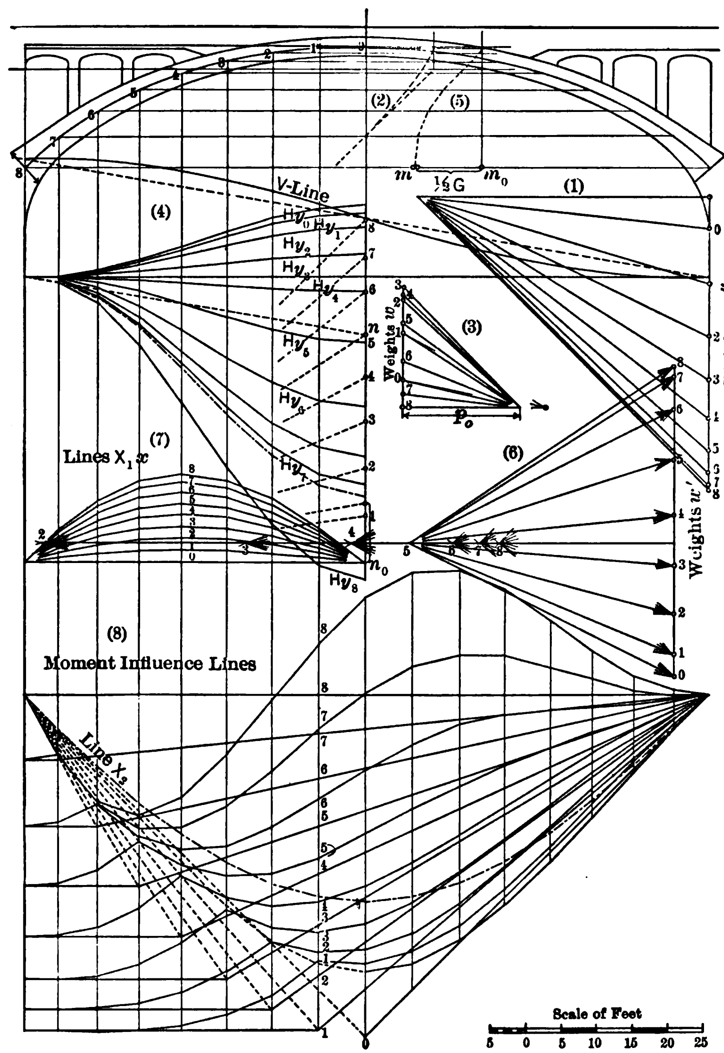

FIG. 43.—Chauderon-Montbenon Bridge at Lausanne. Design by the Graphic Method.

the series of H- and X_1-polygons for the arch-points 0 to 8 are drawn in (4) and (7); this construction is facilitated by the fact that corresponding sides of all the polygons in a series must intersect each other on the closing side. The ordinates of these polygons, measured to twice the scale of lengths, give the moments Hy and X_1x; hence these may be added directly, by means of dividers, to the ordinates of the X_2-line. It remains merely to subtract the combined ordinates from the influence-lines of the moments M_b. The latter, however, are familiarly obtained as triangles; and, in order to have the same scale as for the other influence-lines, the intercept of the triangles at the mid-vertical must be made $=\frac{l}{2}-x=x'$. In this manner are obtained, in (8), the influence lines of the moments M about the points 0 to 8. On account of the close crowding of the oblique lines, the ordinates lying to the left of the respective arch-points are transferred to horizontal axes. The scale to be applied is twice the scale of lengths; i.e., if 1 in.$=k$ linear units, 2 ins.$=k$ moment units.

The axial thrust at any section C (Fig. 15) is given by the equation

$$N=H\cdot\cos\phi+V\cdot\sin\phi.$$

To represent this equation, we use the influence-lines for H and V, the latter being given by $V=V_b+X_1$. In Fig. 43 (4), these influence-lines are drawn.

In calculating the maximum fiber-stresses at any section by

$$f_{1,2}=\frac{N}{A}\mp\frac{M}{Z}$$

where $Z=$the section modulus), the cases of loading considered are those producing the maximum positive or negative values of M. To be precise, it is the moments about the core-points of the section that should be made a maximum or a minimum; nevertheless the difference in the lengths of loading is insignificant, and by the above simplifying assumption the number of cases of loading to be investigated is reduced by one-half.

The following data refer to the reinforced arch-rib of which a strip 3 ft. wide is taken for consideration. The reinforcement consists of 4 angles, $3\times5\times\frac{3}{8}$ ins., giving $A_s=4\times2.9=11.6$ sq.ins. Hence, if d is the depth of section in feet, and assuming the coefficient $n=E_s:E_c=11$ (as specified by the Building Code of Switzerland), we have

$$A=3d+11\cdot A_s=3d+0.89 \text{ (sq.ft.)}.$$

The sectional constants and coordinates of the arch-axis are given in the following table:

TABLE I

($A = A_c + 11 \cdot A_s$; $I = I_c + 11 \cdot I_s$)

Point.	x	y'	d	$\sin\phi$	$\cos\phi$	A	I	$1/I$	W''
0	0	0	2.46	0	1.00	8.27	4.64	.216	.106
1	6.6	0.36	2.49	0.11	0.99	8.36	4.82	.207	.206
2	13.1	1.01	2.56	0.18	0.98	8.55	5.19	.193	.189
3	19.5	2.30	2.74	0.26	0.96	9.09	6.31	.158	.162
4	25.8	3.94	2.82	0.34	0.94	9.35	6.86	.146	.143
5	31.8	6.30	3.05	0.41	0.91	10.05	8.60	.116	.115
6	37.6	9.19	3.48	0.48	0.88	11.33	12.25	.082	.084
7	43.2	12.70	4.30	0.55	0.84	13.80	22.12	.045	.046
8	48.0	16.98	6.10	0.61	0.79	19.19	60.55	.016	.018

The quantities W'', also W' and W in Table II, are calculated from the values of w'', w' and w for each three consecutive points by the formula

$$W = \tfrac{1}{6}(w_{m-1} + 4w_m + w_{m+1}).$$

The position of the axis of abscissæ is determined by

$$y_0 = \frac{\Sigma W''y'}{\Sigma W''} = \frac{3.452}{1.069} = 3.23 \text{ ft.},$$

which checks with the construction, Fig. 43 (2). Having fixed the axis, the ordinates y and the other quantities in Table II may be determined.

TABLE II

Point.	y	y/I	x/I	W	W'	$\dfrac{\cos^2\phi}{A}$
0	3.23	.698	0	.329	0	$\frac{1}{2}$(.12)
1	2.87	.596	1.36	.578	1.33	.12
2	2.22	.427	2.52	.404	2.42	.11
3	0.93	.148	3.09	.148	2.95	.10
4	− 0.71	−.103	2.81	−.104	3.00	.10
5	− 3.07	−.353	3.71	−.311	3.45	.08
6	− 5.96	−.486	3.07	−.459	2.99	.07
7	− 9.46	−.428	1.95	−.408	1.94	.05
8	−13.75	−.228	0.79	−.149	0.59	.03

The correction to be applied to the intercept of the funicular polygon (5), taking $p_0 = 1.46$ as the pole-distance of the force polygon (3), is calculated by

$$c = \frac{1}{p_0}\sum_0^8 \frac{\cos^2\phi}{A} = \frac{1}{1.46} \times 0.72 = 0.49 \text{ ft.};$$

hence one-half of the scale-unit for the H-curve will be $mm_0 = 9.50 + 0.49 = 9.99$ ft., or $G = 2mm_0 = 19.98$ ft.

By means of the moment influence lines, constructed as above described in Fig. 43, the effect of the dead loads and then that of the full live load will be determined.

The dead load consists of: *a*. The weight of the arch strip 3 ft. wide together with the superimposed secondary arches and roadway members. These yield the following loads:

Arch-segment,	0	−1	−2	−3	−4	−5	−6	−7	−8
Weight (lbs.).	10,600	11,700	13,940	24,000	8900	24,800	29,800	16,100	

b. The portion of the weight pertaining to the members of the structure between the two halves of the viaduct. The latter amounts to 3070 lbs. per 1 ft. length of bridge. Since this is carried by an arch rib, 6 ft. wide, on each side, each 3 ft. strip receives a uniform load of $\frac{1}{4} \times 3070 = 768$ lbs. per lineal foot.

The clear distance between the two ribs is 14.6 ft.; each rib being 6 ft. wide, the total width of bridge is $14.6 + (2 \times 6) = 26.6$ ft. On this is carried the live load of 100 lbs. per sq.ft. or $100 \times 26.6 = 2660$ lbs. per lineal foot. For the 3 ft. width of rib, the share of the live load is $\frac{1}{4} \times 2660 = 665$ lbs. per lineal foot.

The thrusts and moments and the resulting fiber stresses produced by these loadings are given for some of the sections in the following table. For the two cases of loading (dead load and full live load), the stresses were also calculated by the analytic method and were found to agree well with the graphic results.

The horizontal thrust produced by temperature variation was calculated by

$$H_t = \frac{E\omega T \cdot l}{2mm_0 \cdot p_0 \cdot \Delta s},$$

with $E = 2{,}000{,}000$ lbs. per sq.in., $\omega = 0.0000068$, $T = \pm 40°$ F., $E\omega T = 78{,}000$ lbs. per sq.ft., $l = 96$ ft., and, in the drawing, $2m_0m = 20$ ft., $p_0 = 1.46$ and the divisions $\Delta s = 6.56$ ft. Hence

$$H_t = \frac{78000 \times 96}{20 \times 1.46 \times 6.56} = 39{,}600 \text{ lbs.}$$

This thrust acts at the height of the axis of abscissæ; therefore the resulting moments are given by $H_t \cdot y$. In the following table these temperature stresses are included and ultimately added to the maximum stresses occurring with the severest live load condition.

TABLE III.—STRESSES

(+ COMPRESSION, — TENSION)

	Condition of Loading.	N (tons)	M (ft.-tons)	$\frac{N}{A}$	$\frac{M}{I}\cdot\frac{d}{2}$	Stress in Concrete: Upper Fiber.	Stress in Concrete: Lower Fiber.
				Pounds per Square Inch.			
Section 0	Unloaded	104	− 8.7	177	− 32	209	145
	Fully loaded	128	− 16.6	217	− 61	278	156
	Critical loading for +M	113	+ 0.5	193	+ 2	191	195
	Critical loading for −M	118	− 25.8	201	− 95	296	106
	Temperature effect	± 20	± 63.9	± 33	±236	∓203	±269
	Max.+temperature	.....	.....	.....	.....	**499**	**−103**
	Min.+temperature	.....	.....	.....	.....	**− 12**	**464**
Section 2	Unloaded	105	− 11.4	172	− 39	211	133
	Fully loaded	129	− 14.6	211	− 50	261	161
	Critical loading for +M	118	+ 7.3	193	+ 25	168	218
	Critical loading for −M	116	− 33.2	190	−114	304	76
	Temperature effect	± 19	± 43.8	± 31	±150	∓119	±181
	Max.+temperature	.....	.....	.....	.....	**423**	**−105**
	Min.+temperature	.....	.....	.....	.....	**49**	**399**
Section 3	Unloaded	107	− 12.0	165	− 36	201	129
	Fully loaded	131	− 10.6	203	− 32	235	171
	Critical loading for +M	123	+ 33.3	191	+101	90	292
	Critical loading for −M	115	− 55.8	178	−169	347	9
	Temperature effect	± 19	± 18.5	± 29	± 55	∓ 26	∓ 84
	Max.+temperature	.....	.....	.....	.....	**373**	**− 75**
	Min.+temperature	.....	.....	.....	.....	**64**	**376**
Section 4	Unloaded	112	+ 4.2	168	+ 12	156	180
	Fully loaded	113	+ 11.8	207	+ 34	173	241
	Critical loading for +M	132	+ 48.2	198	+138	60	336
	Critical loading for −M	118	− 32.3	177	− 92	269	85
	Temperature effect	± 19	∓ 14.0	± 28	∓ 40	± 68	∓ 12
	Max.+temperature	.....	.....	.....	.....	**337**	**73**
	Min.+temperature	.....	.....	.....	.....	**− 8**	**348**
Section 6	Unloaded	121	+ 43.3	150	+ 85	65	235
	Fully loaded	148	+ 53.7	184	+106	78	290
	Critical loading for +M	135	+ 93.1	168	+184	− 16	352
	Critical loading for −M	133	+ 3.9	166	+ 8	158	174
	Temperature effect	± 17	∓118.0	± 22	∓233	±255	∓211
	Max.+temperature	.....	.....	.....	.....	**413**	**− 37**
	Min.+temperature	.....	.....	.....	.....	**−271**	**563**
Section 8	Unloaded	135	+ 9.8	99	+ 7	92	106
	Fully loaded	163	− 16.8	120	− 12	132	108
	Critical loading for +M	146	+ 75.0	107	+ 53	54	160
	Critical loading for −M	153	− 81.9	112	− 58	170	54
	Temperature effect	± 16	∓272.4	± 11	∓192	±203	∓181
	Max.+temperature	.....	.....	.....	.....	**373**	**−127**
	Min.+temperature	.....	.....	.....	.....	**−149**	**341**

The above calculations indicate, under unfavorable combination of temperature and live load, excessive tensile stresses in the concrete near the abutments. We must therefore compute the stresses on the basis of Phase II, in which the elastic coefficient of the concrete is no longer considered constant. Applying this treatment to section 6, assuming a diminished elastic coefficient for the concrete in tension, we obtain the following results:

The distance x of the neutral axis from the compression fiber at this section is obtained by substituting in Eq. (7) the values of N and M produced by the combination of loading and temperature giving maximum tension in the upper fibers, viz.,

$$\begin{cases} N=135-17=118 \text{ tons}, \\ M=93.1+118.0=211.1 \text{ ft.-tons}. \end{cases}$$

Hence $p_n=\frac{211.1}{118}=1.79$ ft.; also $b=3$ ft., $n \cdot A_s=0.89$ sq.ft., $n \cdot I_s=1.893$ ft.^{4} Assuming $m=0.4$, the cubic equation becomes

$$x^3+0.20x^2+30.38x-73.2=0,$$

$$\therefore \quad x=2.083 \text{ ft.}$$

Eq. (7*a*) then gives

$$[\tfrac{1}{2}\times3.0(\overline{2.083}^2-0.4\times\overline{1.397}^2)+0.89\times0.343]f_u=236{,}000.$$

$$\therefore \quad f_u=41{,}700 \text{ lbs./ft.}^2=289 \text{ lbs./in.}^2 \text{ (at 1 ft. from axis)}.$$

The compressive stress in the concrete will then be

$$f_{cp}=289\times2.083=\mathbf{602} \text{ lbs./sq.in.};$$

the tensile stress

$$f_{ct}=0.4\times289\times1.397=\mathbf{161} \text{ lbs./sq.in.}$$

The stresses in the steel become

$$\begin{cases} f_{sp}=11\times289\times1.95=6200 \text{ lbs./sq.in.}, \\ f_{st}\;=11\times289\times1.26=4010 \text{ lbs./sq.in.} \end{cases}$$

If we do not count upon any tensile resistance of the concrete, Eq. (7) with $m=0$ gives $x=1.578$ ft.; the maximum compression in the concrete will then be $f_{cp}=732$ lbs./sq. in.; the tension in the steel $f_{st}=9040$ lbs./sq.in.

On account of the transfer of part of the weight of the arch directly to the steel reinforcement during erection, a fact not considered in the above computations, the concrete stresses are somewhat lower while the tension in the steel is somewhat higher than the values found above.

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See also footnotes in the text.

KEY TO PLATE I

CHART FOR THE DESIGN OF CONCRETE SECTIONS WITH DOUBLE REINFORCEMENT

m, m' = coefficients of resisting moment of the section; defined by $M=m\cdot bd^2\cdot f_c$, if the concrete is stressed to the limit, or by $M=m'\cdot bd^2\cdot f_c$, if the steel is stressed to the limit;

f_c = unit compression in extreme upper concrete fibers (pounds per square inch);

f_s = unit tension in lower steel reinforcement (pounds per square inch);

b = width of section (inches);

d = total depth of section (inches);

x = distance of neutral axis below extreme upper fibers, expressed as a fraction of d;

For *single* reinforcement, use points on axis of abscissæ.

For *positive* moments (compression in upper fibers), use *black* curves and figures.

For *negative* moments (compression in lower fibers), use *red* curves and figures.

For sections subject to *reversal* of moment, use intersection of corresponding black and red curves.

Line RR = locus of points on curves of same color giving minimum combined reinforcement.

NOTE.—When more convenient, the black curves may be used instead of the red curves upon interchanging the words " upper " and " lower " throughout key and chart.

NUMERICAL EXAMPLES

ILLUSTRATING USE OF PLATE I

EXAMPLE 1.—*To design a section for a* POSITIVE *bending moment.*

Given $M=480{,}000$ in.-lbs., $b=12$ ins., $d=20$ ins., max. $f_c=500$ lbs. per square inch.

With these values, $$m=\frac{M}{bd^2\cdot f_c}=\frac{480000}{12\times 400\times 500}=+0.20.$$

The coordinates of *any* point of the curve $m=+0.20$ will give a suitable combination of upper and lower reinforcement to resist the above bending moment. Thus the respective percentages may be made:

upper		lower	
0	and	3.1	(i.e., single reinforcement);
0.25	and	2.0	
0.62	and	1.22	(see Example 3);
0.8	and	1.0	(most economical);
1.0	and	0.83	(both stresses limiting).

With the last combination, the tension reinforcement would have the full allowable stress of $30 \cdot f_c = 15{,}000$, so that a further reduction of the tension reinforcement is impossible. A further increase in the compression reinforcement will not augment the resisting moment of the section unless the allowable stress in the tension steel is raised.

The most economical combination (0.8 and 1.0 per cent) is obtained by taking the point on the line *RR*. For this combination (by interpolation between the dotted lines), $x=0.336$ and $f_s : f_c = 25.23$. Hence $f_s = 25.23 f_c = 12{,}615$ lbs. per square inch, tension in lower reinforcement. The compression in the upper reinforcement is simply $n(=15)$ times the stress in the concrete at the same point. The tensile resistance of the concrete below the neutral axis is neglected.

EXAMPLE 2.—*To design a section for a* NEGATIVE *bending moment.*

Given $M=-360{,}000$ in.-lbs., $b=12$ ins., $d=20$ ins., max. $f_c=500$ lbs. per square inch.

With these values, $$m=\frac{-360{,}000}{12\times400\times500}=-\mathbf{0.15}.$$

The coordinates of *any* point of the curve $m=-\mathbf{0.15}$ (*red* curve) will give a suitable combination of upper and lower reinforcement to resist the above bending moment. Thus the respective percentages may be made:

upper		lower	
0.99	and	0	(i.e., single reinforcement);
0.80	and	0.17	(most economical combination);
0.62	and	0.38	(both stresses limiting);
0.62	and	1.22	(see Example 3);
0.62	and	*any* percentage >0.38.	

The same values may also be obtained by using the *black* curve ($m=+0.15$) while imagining the section turned around, i.e., reading the *upper* reinforcement on the horizontal axis and the *lower* reinforcement on the vertical axis. The intersection of the black curve with the line *RR* gives the most economical combination, 0.80 and 0.17 per cent. For this combination, $x=0.347$ (measured from the *bottom* of the section) and $f_s : f_c = 23.87$. Hence $f_s = 23.87\times500 = 11{,}935$ lbs. per square inch tension in *upper* reinforcement.

EXAMPLE 3.—*To design a section to take* REVERSAL *of bending moment.* (Combination of Examples 1 and 2.)

Given maximum *positive* $M=480{,}000$ lbs. and maximum *negative* $M=-360{,}000$ lbs., $b=12$ ins., $d=20$ ins., max. $f_c=500$ lbs. per square inch, max. $f_s=15{,}000$ lbs. per square inch.

With these values, $m=\dfrac{M}{bd^2\cdot f_c}=+0.20$ and -0.15, respectively. The *black* curve $m=+0.20$ and the red curve $m=-0.15$ intersect in the point (1.22, 0.62), indicating the requirement of 0.62 per cent upper reinforcement and 1.22 per cent lower reinforcement to exactly resist the above bending moments. Both strength and economy may be increased, however, by moving along the curve $m=+0.20$ to the intersection of the *RR* line, giving 0.8 per cent upper and 1.0 per cent lower reinforcement. By this change, notwithstanding the reduced reinforcement, the negative resisting moment of the section is increased from $m=-0.15$ to $m=-0.196$.

The point (1.0, 0.8) yields the ratio $f_s:f_c=25.23$. Hence, under the maximum positive moment,

$$f_c=500 \text{ (upper fiber),}$$

and $$f_s=25.23\cdot f_c=12{,}615 \text{ (lower reinforcement).}$$

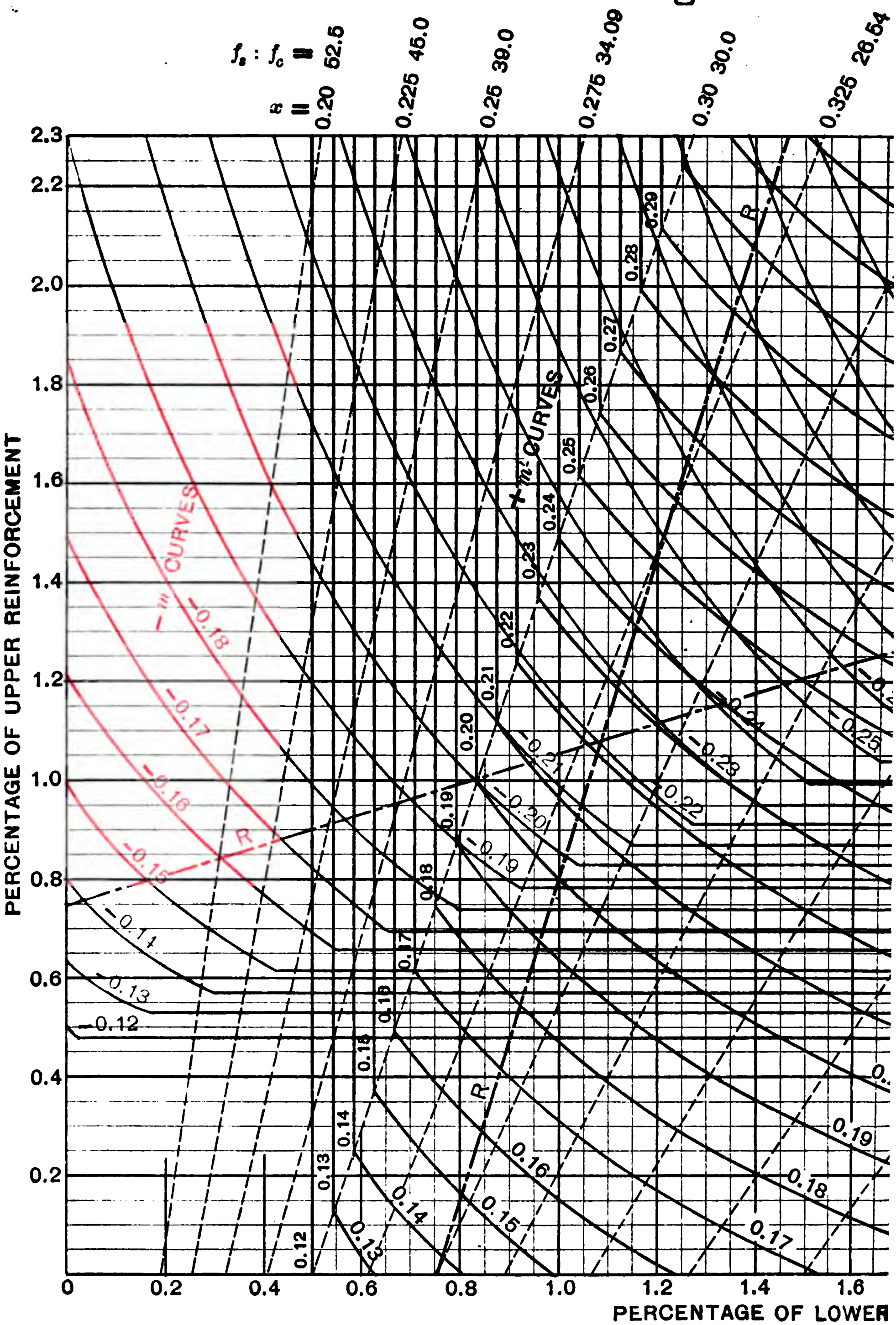
The point (0.8, 1.0) yields the ratio $f_s:f_c=30.90$. Hence, for maximum negative moment, since the full resisting moment of the section is not utilized,

$$f_s=\frac{.15}{.196}\times 15{,}000=11{,}490 \text{ (upper reinforcement),}$$

and $$f_c=f_s\div 30.90=372 \text{ (lower fiber).}$$

NOTE.—If no compression reinforcement is used, the chart shows that 0.5 per cent of tension reinforcement gives a perfectly balanced section (defined by $f_s:f_c=30$). The chart also shows that when the required single reinforcement exceeds 0.75 per cent, the introduction of compression reinforcement is conducive to economy even though the section is not subject to reversal of bending moment.

PLATE I.—Chart for the Design of Conc



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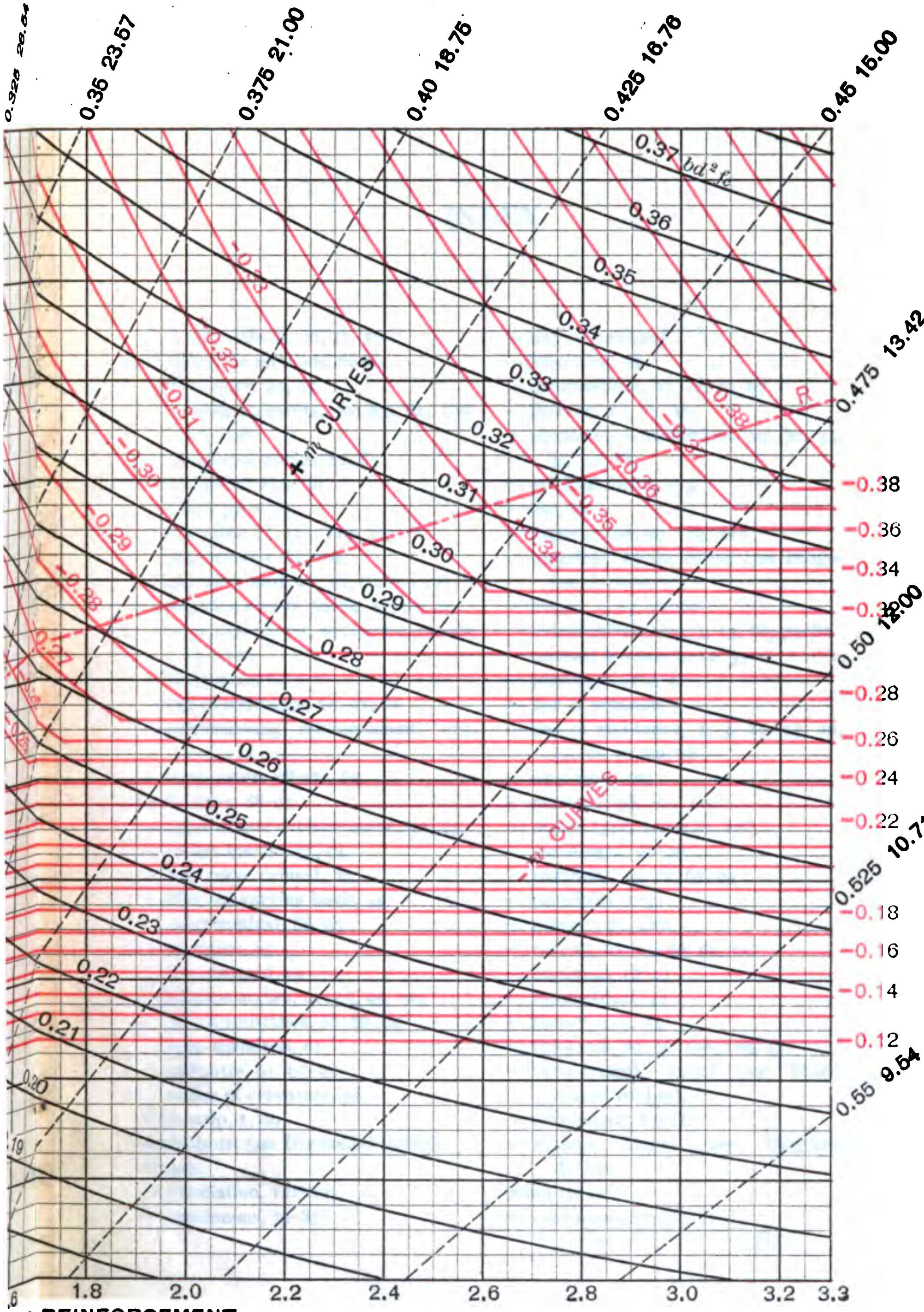

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